EE 503 – Week 10 [31-October-2013]

This week: Dr. Osonde Osoba

- Stochastic Convergence Review
- Moment-generating functions, Characteristic functions
  - Definition, Properties
  - Examples
  - Levy's Continuity Theorem
- Central Limit Theorem
  - Proof
  - Generalizations: Lindeber-Feller CLT, SαS-CLT
  - Confidence Intervals for M/P ← next week

H/W

Gubner : 6.27, 6.28 ← next week

L-G : 4.102, 4.106

5.99 – 5.102

7.23 – 7.29

8.39 – 8.41 ← next week

I Moment-Generating and Characteristic functions:

X : rv with pdf $f_X(x)$

Define the moment-generating function $M_X(s)$ for X:

$$\boxed{M_X(s) = E_X[\exp(sX)]}$$

$\forall s \in \mathbb{R}$ for which $M_X(s)$ exists

$$M_X(s) = \int \exp(sx) \cdot f_X(x)\, dx \qquad [X \text{ continuous}]$$

$$M_X(s) = \sum_k \exp(sk) \cdot p_X(k) \qquad [X \text{ discrete}]$$

[Similar to a Laplace transform.]

Define the characteristic function $\phi_X(t)$ for X:

$$\boxed{\phi_X(t) = E_X[\exp(itX)]} \qquad \begin{array}{l} i = \sqrt{-1} \\ t \in \mathbb{R} \end{array}$$

So $\phi_X(t) = M_X(s)\big|_{s=it}$

[Similar to a Fourier transform.]

$M_X(s)$ may not exist if the integral/sum is non-convergent.

$\phi_X(t)$ always exists since: $\forall t \in \mathbb{R}$

$$|\phi_X(t)| \leq E_X[|\exp(itX)|] = E_X[1] = 1$$

The name "moment-generating function" comes from the fact that all moments $E[X^k]$ are expressible in terms of $M_X(s)$ or $\phi_X(t)$. This is an effect of the properties of the (complex) exponential:

$$\exp(s\cdot X) = \sum_{k=0}^{\infty} \frac{(s\cdot X)^k}{k!}$$

$$= 1 + s\cdot X + \frac{s^2 X^2}{2} + \frac{s^3 X^3}{3!} + \cdots$$

if abs convergent sum
($\Leftrightarrow M_X(s)$ exists)

$$\Rightarrow E_X[\exp(s\cdot X)] = \sum_{k=0}^{\infty} \frac{s^k E[X^k]}{k!} = 1 + s\,E[X] + \frac{s^2 E[X^2]}{2} + \cdots$$

Theorem:

(a) $$E[X^k] = \left.\frac{d^k M_X(s)}{ds^k}\right|_{s=0}$$

(b) $$i^k \cdot E[X^k] = \left.\frac{d^k \phi_X(t)}{dt^k}\right|_{t=0}$$

if $E[|X|^k] < \infty$

$$\therefore\ E[X^0] = E[1] = 1 = \phi_X(0) = M_X(0)$$

Properties of $\phi_X(t)$:

1/ (a) $\phi_X(0) = 1$

(b) $|\phi_X(t)| \leq 1$

2/ [Uniqueness + Inversion]

(a) $X \stackrel{d}{=} Y \iff \phi_X(t) = \phi_Y(t) \quad \forall t \in \mathbb{R}$

(b) $f_X(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(-itx)\, \phi_X(t)\, dt$

[cf. Fourier inversion]

3/ $Y = aX + b \implies \phi_Y(t) = \exp(itb) \cdot \phi_X(at)$

$\because E_X[\exp(it(aX+b))] = \exp(itb) \cdot E_X[\exp(iatX)]$

$= \exp(itb) \cdot \phi_X(at)$

4/ [Sums] (also for $m_Z(s)$)

$\left.\begin{array}{l} Z = X + Y \\ X, Y \text{ independent} \end{array}\right\} \iff \phi_Z(t) = \phi_X(t) \cdot \phi_Y(t)$

Proof: ($\Rightarrow$) $\phi_Z(t) = E[\exp(it(X+Y))] = E[\exp(itX) \cdot \exp(itY)]$

$\stackrel{ind}{=} E[\exp(itX)] \cdot E[\exp(itY)]$

$= \phi_X(t) \cdot \phi_Y(t)$ //

($\Leftarrow$) by uniqueness

$$S_n = \sum_{i=1}^{n} X_i \qquad \{X_i\} \text{ iid}$$

$$\Rightarrow \phi_{S_n}(t) \overset{\text{by 4.}}{=\!=} \prod_{i=1}^{n} \phi_{X_i}(t)$$

$$\overset{\text{by 2a.}}{=\!=} \prod^{n} \phi_X(t)$$

$$\phi_{S_n}(t) = (\phi_X(t))^n \;/\!/$$

$$\bar{X}_n = 1/n \sum_{i=1}^{n} X_i = 1/n \cdot S_n$$

$$\Rightarrow \phi_{\bar{X}_n}(t) \overset{\text{by 3.}}{=\!=} \phi_{S_n}(t/n)$$

$$\phi_{\bar{X}_n}(t) = [\phi_X(t/n)]^n \;/\!/$$

Examples of $M_X(s)$ and $\phi_X(t)$:

a. $X \sim b(n,p)$

$$M_X(s) = E[\exp(s \cdot X)]$$

$$= \sum_{k=0}^{n} \exp(s \cdot k) \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot e^{sk} \cdot p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (p \cdot e^s)^k (1-p)^{n-k}$$

$$M_X(s) \overset{\text{Bin Thm}}{=\!=} (1-p+p \cdot e^s)^n$$

[always exist because finite sums of finite #'s are abs. convergent] recall that $X \sim b(n,p) \Rightarrow X = \sum_{i=1}^{n} Y_i$

where $Y_i \sim \text{Bernoulli}(p)$, $\{Y_i\}_i$ iid

$$\Rightarrow \phi_{Y_i}(t) = \phi_Y(t) = (1-p)\cdot \exp(s\cdot(0)) + p\cdot \exp(s\cdot 1)$$

$$\phi_Y(t) = 1-p + p\cdot e^s \qquad \forall i$$

$$\Rightarrow \quad \phi_X(t) = [\phi_Y(t)]^n$$

$$= (1-p+pe^s)^n$$

b) $X \sim \exp(\theta)$

$$\phi_X(t) = E_X[\exp(itX)]$$

$$= 1/\theta \int_0^\infty \exp(itx)\cdot \exp(-x/\theta)\, dx$$

$$= 1/\theta \int_{\mathbb{R}^+} \exp(x(it - 1/\theta))\, dx$$

$$= \frac{1}{\theta(it - 1/\theta)} \cdot \exp(x(it - 1/\theta)) \Big|_0^\infty$$

$$= \frac{-1}{\theta it - 1}$$

$$\therefore \quad \phi_X(t) = \frac{1}{1 - \theta it}$$

Recall if $X_i \sim \exp(\theta)$ iid

$$Z = \sum_i^n X_i \implies Z \sim n\text{-Erlang}(\theta) = \gamma(\alpha = n, \theta)$$

$$\Rightarrow \phi_Z(t) = [\phi_X(t)]^n = \frac{1}{(1 - i\theta t)^n}$$

c: $\{N, \{X_i\}_i\}$: independent
$\{X_i\}_i$: iid
N : discrete, ≥ 0

$S_N = \sum_{i=1}^{N} X_i$
[Doubly random sum]
[Compound process]

$$\phi_{S_N}(t) = E[\exp(itS_N)]$$
$$\stackrel{\text{IE}}{=} E_N E_{S_N|N}[\exp(itS_N)|N]$$
$$\stackrel{\Sigma}{=} \sum_n p_N(n) \cdot E_{S_N|N}[\exp(itS_n)|N=n]$$
$$\stackrel{\text{iid/ind}}{=} \sum_n p_N(n) \left(E_X[\exp(itX)]\right)^n$$
$$= \sum_n p_N(n) (\phi_X(t))^n$$

$$\phi_{S_N}(t) = E_N[(\phi_X(t))^N]$$

note: $i \cdot \phi'_{S_N}(t)\Big|_{t=0} = \left[i \cdot \phi'_X(t) \cdot \left(\sum_n p_N(n) \cdot n \cdot (\phi_X(t))^{n-1}\right)\right]\Big|_{t=0}$

i.e $E[S_N] = \mu_X \cdot \mu_N$ [recall Wald's Identity] d/ $X \sim \text{Poisson}(\lambda) \iff f_X(k) = e^{-\lambda} \cdot \lambda^k / k!$

$$\Rightarrow \phi_X(t) = E[\exp(itX)]$$
$$= \sum_k \exp(itk) \cdot e^{-\lambda} \cdot \lambda^k / k!$$
$$= e^{-\lambda} \sum_{k=0}^{\infty} (\lambda \exp(it))^k / k!$$
$$= e^{-\lambda} \cdot \exp(\lambda \cdot \exp(it))$$
$$= \exp[\lambda(\exp(it) - 1)]$$

Suppose $N \sim \text{Poisson}(\lambda)$, $\{X_k\}_{k \geq 0}$ iid, indep of N

$S_N = \sum_{k=0}^{N} X_k$ [Compound poisson process]

$$\Rightarrow \phi_{S_N}(t) = \sum_{n \geq 0} P_N(n) \cdot E_{S_N|N}\left[\exp\left(it \sum_{k=0}^{n} X_k\right) \middle| N = n\right]$$
$$= e^{-\lambda} \sum_{n \geq 0} \frac{\lambda^n}{n!} \prod_{k=0}^{n} E_{X_k}[\exp(itX_k)]$$
$$= e^{-\lambda} \sum_{n \geq 0} \frac{\lambda^n \cdot (\phi_X(t))^n}{n!}$$
$$= e^{-\lambda} \sum_{n \geq 0} \frac{(\lambda \cdot \phi_X(t))^n}{n!}$$
$$= e^{-\lambda} \cdot \exp(\lambda \cdot \phi_X(t))$$
$$\phi_{S_N}(t) = \exp[\lambda(\phi_X(t) - 1)]$$

e: $X \sim N(\mu, \sigma^2) \Rightarrow \phi_X(t) = \exp\left[i\mu t - \frac{1}{2}\sigma^2 t^2\right]$

Proof:

$$f_X(x) = (2\pi\sigma^2)^{-1/2} \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$\phi_X(t) = E_X[\exp(itX)] = \int f_X(x) \cdot \exp(itx)\, dx$$
$$= (2\pi\sigma^2)^{-1/2} \cdot \int \exp\left[itx - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$$
$$= C \cdot \int \exp\left[it(\sigma z + \mu) - z^2/2\right] dz$$

$$C = \sigma \cdot (2\pi\sigma^2)^{-1/2} = \left(\sqrt{2\pi}\right)^{-1}$$

$$\Rightarrow \phi_X(t) = C \cdot \exp(it\mu) \cdot \int \exp\left[it\sigma z - z^2/2\right] dz$$

$$it\sigma z - z^2/2 = -\frac{1}{2}\left[z^2 - 2it\sigma z + (it\sigma)^2 - (it\sigma)^2\right]$$

$$\Rightarrow it\sigma z - z^2/2 = -\frac{1}{2}\left[(z - it\sigma)^2 + t^2\sigma^2\right]$$

$$\Rightarrow \phi_X(t) = C \cdot \exp(it\mu) \cdot \exp(-t^2\sigma^2/2) \cdot \int \exp\left(-\frac{1}{2}(z - it\sigma)^2\right) dz$$

$$r = z - it\sigma \Rightarrow dr = dz$$

$$\Rightarrow \phi_X(t) = \exp(it\mu - t^2\sigma^2/2) \cdot \overbrace{\underbrace{\frac{1}{\sqrt{2\pi}} \int \exp(-r^2/2)\, dr}_{\int f_Z(z)\, dz \text{ for } Z \sim N(0,1)}}^{=1}$$

$$\therefore \boxed{\phi_X(t) = \exp(it\mu - t^2\sigma^2/2)}$$

Recall $\vec{X} \in \mathbb{R}^d$

$\vec{X} \sim N(\vec{\mu}, \underline{K}_{xx})$

$$\Leftrightarrow \quad f_{\vec{X}}(\vec{x}) = \frac{\exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu})^T \underline{K}_{xx}^{-1} (\vec{x}-\vec{\mu})\right]}{\sqrt{(2\pi)^d \, \mathrm{Det}(K_{xx})}}$$

$$\Leftrightarrow \quad \boxed{\begin{aligned} \phi_{\vec{X}}(\vec{t}) &= E_{\vec{X}}[\exp(i\,\vec{t}^T\vec{X})] = E_{\vec{X}}[\exp(i\langle \vec{t}, \vec{X}\rangle)] \\ &= \exp\left[i\langle \vec{t}, \vec{\mu}\rangle - \tfrac{1}{2}\langle \vec{t}, \underline{K}_{xx}\vec{t}\rangle\right] \end{aligned}}$$

[An example of a multi-dimensional ch.f.]

Kac's Theorem: $\{X_i\}_{i=1}^n$ independent

$$\Leftrightarrow \quad \phi_{\vec{X}}(\vec{t}) = \prod_{i=1}^{n} \phi_{X_i}(t_i) \quad //$$

Note that $f_{\vec{X}}$ for the Gaussian r. vector is a function of $\underline{K}_{xx}^{-1}$ (inverse covar. mtx). $\underline{K}_{xx}^{-1}$ does not always exist (when $\exists i: \lambda_i = 0$ and $\mathrm{Det}(K_{xx}) = 0$). i.e there are Gaussian distributions without well-defined pdfs. The Ch.f. always exists though. Since $\phi_{\vec{X}}(\vec{t})$ only involves $\underline{K}_{xx}$. There are many other distributions with $\phi_X(t)$ but no $f_X(x)$.

Infinite - Divisibility :

X is infinitely divisible if

$\iff \forall n \in \mathbb{N}$ (!!!)

$\exists$ iid $\{Y_i\}_{i=1}^n$:

$$\boxed{X \overset{d}{=} \sum_{i=1}^{n} Y_i}$$

$\iff \forall n \in \mathbb{N}$

$\exists$ a unique ch.f. $\phi_Y(t)$:

$$\boxed{\phi_X(t) = [\phi_Y(t)]^n}$$

e.g. (a) $X \sim \text{Poisson}(\lambda)$

$\forall n \in \mathbb{N}$, $Y_i \sim \text{Poisson}(\lambda/n)$

$$\phi_X(t) = \exp(\lambda(e^{it}-1))$$

$$\phi_{Y_i}(t) = \exp(\lambda/n\,(e^{it}-1))$$

$$\Longrightarrow \phi_X = (\phi_Y)^n$$

$\therefore$ X is infinitely divisible //

(b) $X \sim N(\mu, \sigma^2)$

$\forall n \in \mathbb{N}$, let $Y_i \sim N(\mu/n, \sigma^2/n)$

$$\exp(it\mu - t^2\sigma^2/2) = [\exp(it\mu/n - t^2\sigma^2/2n)]^n$$

$\therefore$ X is infinitely divisible e.g.: $X_i \sim N(\mu, \sigma^2)$

$$\Rightarrow \begin{cases} \left(\sum_{i=1}^{n} X_i\right) \sim N(n\mu, n\sigma^2) \\ \left[\sqrt{n}\, X + \mu_x(n-\sqrt{n})\right] \sim N(\mu n, n\sigma^2) \end{cases}$$

i.e. $c_n = \sqrt{n}$; $d_n = \mu_x(n-\sqrt{n})$

or use $\mu = 0$

$X_i \sim C(0, d)$

$\phi_x(t) = \exp(itm) \cdot \exp(-d|t|)$

$$\Rightarrow \begin{cases} \left(\sum_{i=1}^{n} X_i\right) \sim C(0, nd) \\ nX \sim C(0, nd) \end{cases}$$

i.e. $c_n = n$; $d_n = 0$

NB: $\Rightarrow \bar{X}_n \stackrel{d}{=} X \quad \forall n$ when $X \sim C$

The class of Symmetric Alpha-stable distribution satisfy this definition.

$X \sim S\alpha S$

$$\Longleftrightarrow \boxed{\phi_X(\omega) = e^{-d|\omega|^{\alpha}}} \qquad \alpha \in (0, 2]$$

$\alpha = 2 \Longleftrightarrow X \sim N(0, d)$

$\alpha = 1 \Longleftrightarrow X \sim C(0, d)$

$E_x\left[|X|^f\right] < \infty$ if $f < \alpha$

(c) $X \sim \gamma(\alpha, \theta)$

$Y_i \sim \gamma(\alpha/n, \theta)$

$\phi_X(t) = (1 - it\theta)^{-\alpha}$

$\phi_{Y_i}(t) = (1 - it\theta)^{-\alpha/n}$

$\Rightarrow \quad (\phi_{Y_i})^n = \phi_X$

$\therefore \quad \gamma(\alpha, \theta)$ is infinitely divisible

Levy processes are based on sums of infinitely divisible rvs $\begin{pmatrix} N(\mu, \sigma^2) \\ \text{Poisson}(\lambda) \end{pmatrix}$.

A special case of infinite divisibility addresses the limiting distribution of iid sums:

$$\left(\sum_{i=1}^{n} X_i\right) \stackrel{d}{=} c_n \cdot X + d_n$$

This special case is called <u>stability</u>:

X is <u>stable</u>:

$\Longleftrightarrow \quad \exists \; c_n, d_n \in \mathbb{R} \;\; \forall \; n \in \mathbb{N}$:

$$\boxed{\sum_{i=1}^{n} X_i \stackrel{d}{=} c_n X + d_n}$$ for iid X_i

where X and X_i have <u>the same</u> distribution.

$$\Longleftrightarrow \quad \boxed{[\phi_X(t)]^n = \exp(i d_n t)\; \phi_X(c_n t)}$$

$d_n \equiv 0 \Leftrightarrow$ strict stability ; $\quad c_n \propto n^{1/\alpha}$

α: index of stability $\in (0, 2]$

Levy's Continuity Theorem :

$\{X_n\}_{n\geq 1}$: with CDF $\{F_n\}_{n\geq 1}$

a/ $$\boxed{X_n \xrightarrow{d} X \implies \phi_{X_n}(t) \longrightarrow \phi_X(t) \quad \forall t \in \mathbb{R}}$$

b/ if $\exists$ a function $\phi_X(t)$:

$$\boxed{\left.\begin{array}{l} \phi_{X_n}(t) \longrightarrow \phi_X(t) \quad \forall t \in \mathbb{R} \\ \phi_X(t) \text{ continuous @ } t=0 \end{array}\right\} \implies X_n \xrightarrow{d} X}$$

where $F_X(x) = \mathcal{F}^{-1}\{\phi_X(t)\}$

Thus we can prove $X_n \xrightarrow{d} X$ by proving that the ch.f's $\phi_n(t)$ converge pointwise to $\phi(t)$.

e.g : [Poisson Law]

$$\left[X_n \sim b(n,p)^{\,n\to\infty,\ p\to 0}\right] \implies X_n \xrightarrow{d} X \sim \text{Poisson}(\lambda = np)$$

proof :

$$\phi_X(t) = \exp\left(\lambda(e^{it}-1)\right)$$

$$\phi_{X_n}(t) = (1-p+pe^{it})^n$$

$$\phi_{X_n}(t) = \left(1 + \frac{np(e^{it}-1)}{n}\right)^n \quad \left[\approx \left(1+\frac{x}{n}\right)^n \to e^x\right]$$

$$= \left(1 + \frac{\lambda(e^{it}-1)}{n}\right)^n \longrightarrow \exp\left[\lambda(e^{it}-1)\right]$$

i.e $\phi_{X_n}(t) \longrightarrow \phi_X(t) \ \forall t \ \therefore X_n \longrightarrow X \sim \text{Poisson}(np)$

Central Limit Theorem:

- $\{X_i\}_{i\geq 1}$: iid, $\sigma_X^2 < \infty$
- $Z_n = STD(\bar{X}_n) = \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} =$

$$= STD\left(\sum_i^n X_i\right) = \frac{\sum^n X_i - n\mu}{\sqrt{n}\,\sigma}$$

$$\Longrightarrow \quad \boxed{Z_n \xrightarrow{d} Z \sim N(0,1)}$$

Proof: Show $\phi_{Z_n} \xrightarrow{e} \phi_Z$ then use Levy's Theorem

$$\phi_n(t) = E[\exp(itZ_n)]$$

$$= E\left[\exp\left[\tfrac{it}{\sqrt{n}\sigma}\cdot\left(\sum^n X_i - n\mu\right)\right]\right]$$

$$= E\left[\exp\left[\frac{it}{\sigma\sqrt{n}}\sum^n (X_i-\mu)\right]\right]$$

$$= E\left[\prod_{i=1}^n \exp\left[\frac{it}{\sigma\sqrt{n}}(X_i-\mu)\right]\right]$$

$$\overset{ind}{=} \prod_{i=1}^n E\left[\exp\left[\frac{it}{\sigma\sqrt{n}}(X_i-\mu)\right]\right]$$

$$\overset{id}{=} \left(E\left[\exp\left[\frac{it}{\sigma\sqrt{n}}(X-\mu)\right]\right]\right)^n$$

$\exp(y) = \sum_{k=0}^{\infty} y^k/k!$

$$\Rightarrow \quad \phi_n(t) = \left(E\left[\sum_{k=1} y^k/k!\right]\right)^n\Bigg|_{y=\frac{it}{\sigma\sqrt{n}}(X-\mu)}$$

$$= \left(E\left[1 + \frac{it(X-\mu)}{\sigma\sqrt{n}} + \frac{(it)^2(X-\mu)^2}{2n\sigma^2} + H.O.T.\right]\right)^n$$

$E[(X-\mu)] = \mu - \mu = 0 \,; \quad E[(X-\mu)^2] = \sigma^2$

$$\Rightarrow \phi_n(t) = \left(1 + 0 + \frac{\sigma^2 (it)^2}{2\sigma^2 n} + E[H.O.T.]\right)^n$$

$$\phi_n(t) = \left(1 - \frac{t^2/2}{n} + E[HOT]\right)^n$$

$E[HOT] = o(t^2/n)$ i.e $E[HOT] \ll t^2/n$ as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} \phi_n(t) \cong \lim_{n\to\infty} \left(1 - \frac{t^2/2}{n}\right)^n$$

$$= \exp(-t^2/2) = \phi(t)$$

$\therefore \quad Z_n \xrightarrow{d} Z \sim N(0,1)$ //

Lindeberg – Feller CLT :

Suppose $\{X_i\}_{i\geq 1}$ is not iid, just independent, $\sigma_i^2 < \infty \ \forall i$

And i/ $s_n^2 = Var\left(\sum_i^n X_i\right) = \sum_i^n \sigma_i^2$

ii/ $\dfrac{\max_i \sigma_i^2}{s_n^2} \longrightarrow 0$

$$\Rightarrow \boxed{\left(\frac{\sum_{i=1}^n (X_i - \mu_i)}{s_n^2}\right) \xrightarrow{d} Z \sim N(0,1)}$$ //

SαS – CLT :

$\{X_i\}_i$: iid SαS ; i.e $\phi_x(\omega) = \exp(-d\,|\omega|^{\alpha})$, $\alpha \in (0,2]$

$$\Rightarrow \boxed{\left(\frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} X_k\right) \xrightarrow{d} Z_\alpha \sim S\alpha S(\alpha, d)}$$

e.g $\frac{1}{n} \sum X_i \quad X_i \sim c(m, d)$