This Week : Dr. Osonde Osoba

- Review
  - Logic, Proof by truth tables
  - Probability Spaces $(\Omega, \mathcal{A}, P)$
- Conditional probability
  - Definition
  - Independence
  - Total Probability, Bayes Theorem
  - Multiplication Theorem
    - Proof by Induction
- Continuity of Probability
  - Borel-Cantelli Lemma I

I/ Proof by Truth tables:

Recall material implication: $p \Rightarrow q$

p	q	$p \Rightarrow q$
1	1	1
0	1	1
1	0	0
0	0	1

$[p \Rightarrow q]$ is logically equivalent to

i/ $[\sim p \vee q]$

ii/ $[\sim q \Rightarrow \sim p]$

How to prove these?

Ex 1 claim i/ $[p \Rightarrow q] \Longleftrightarrow [\sim p \vee q]$

ii/ $[p \Rightarrow q] \Longleftrightarrow [\sim q \Rightarrow \sim p]$

proof: Use a truth table to show that there is no possible combination of truth values for p and q such that each equivalence is false

p	q	$[p \Rightarrow q]$	$\Longleftrightarrow$	$[\sim p$	$\vee$	$q]$	$[p \Rightarrow q]$	$\Longleftrightarrow$	$[\sim q$	$\Rightarrow$	$\sim p]$
1	1	1	1	0	1	1	1	1	0	1	0
0	1	1	1	1	1	1	1	1	0	1	1
1	0	0	1	0	0	0	0	1	1	0	0
0	0	1	1	1	1	0	1	1	1	1	1

QED

Other equivalences include:

$p \Longleftrightarrow \sim\sim p$, [Double negation], $\sim(p \vee q) \Longleftrightarrow (\sim p \wedge \sim q)$, [De Morgan's], etc

We can also prove the validity of rules of inference (argument forms) using the same method.

Ex2 [Modus Ponens]

$$\left.\begin{array}{l} p \Rightarrow q \\ \underline{p} \\ \therefore q \end{array}\right\} \quad \text{or} \quad [(p \Rightarrow q) \wedge p] \Rightarrow q$$

Proof:

p	q	$[(p$	$\Rightarrow$	$q)$	$\wedge$	$p]$	$\Rightarrow$	q
1	1	1	1	1	1	1	1	1
0	1	0	1	1	0	0	1	1
1	0	1	0	0	0	1	1	0
0	0	0	1	0	0	0	1	0

Ex 3: Prove

i) $p \Rightarrow q$, $\underline{\sim q}$, $\therefore \sim p$ — Modus Tollens

ii) $p \vee q$, $\underline{\sim p}$, $\therefore q$ — disjunctive Syllogism

iii) $p \Rightarrow q$, $\underline{q \Rightarrow r}$, $\therefore p \Rightarrow r$ — Hypothetical Syllogism

Proof: Truth Tables.

II Conditional Probability:

Review:

Events are measurable sets, i.e. elements of a sigma algebra. σ-algebras are CUT. Probability measures P are CAT functions $P: \mathcal{Q} \longrightarrow [0,1]$

given $A, B \in \mathcal{Q}$ and $P(B) \neq 0$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

$P(A|B) \approx \text{degree}(B \Rightarrow A)$

Qu is $(A \cap B) \in \mathcal{Q}$ if $A, B \in \mathcal{Q}$? Why?

Independence:

A, B are independent

$\iff P(A \cap B) = P(A) \cdot P(B)$

$\iff P(A|B) = P(A)$

The Event $\{A_k\}_k$ are independent

$\iff P(\bigcap_k A_k) = \prod_k P(A_k)$

Independence $\neq$ Mutual Exclusivity

A, B mutually exclusive $\iff A \cap B = \phi$

Questions:

i) $[A \cap B = \phi] \overset{???}{\iff} [P(A \cap B) = 0]$

ii) $[P(E) = 0] \overset{??}{\implies} [E = \phi]$

iii) Can an event be independent of itself ?

iv) $[E \text{ not measurable}] \overset{??}{\iff} [P(E) = 0]$

Ex 4 Given a probability space $(\Omega, \mathcal{Q}, P)$ and a measurable set B with $P(B) \neq 0$. Define the reduced σ-alg

$B \cap \mathcal{Q} = \{B \cap E \mid E \in \mathcal{Q}\}$

Is $(B \cap \Omega, B \cap \mathcal{Q}, P(\cdot|B))$ also a probability space?

Ans: Show i) $P(\cdot|B)$ is CAT and ii) $B \cap \mathcal{Q}$ is CUT

i) I : $P(B|B) = P(B \cap B)/P(B) = P(B)/P(B) = 1$ //

CA : $\{A_i\}_i$: countable collection of disjoint events.

$$\Rightarrow P(\bigcup_i A_i | B) = \frac{P((\bigcup_i A_i) \cap B)}{P(B)}$$

$$= \frac{P(\bigcup_i (A_i \cap B))}{P(B)} = \sum_i \frac{P(A_i \cap B)}{P(B)}$$

$$= \sum_i P(A_i | B)$$ //

ii) $\underline{\underline{C}}$: $S \in B \cap \mathcal{Q} \Rightarrow S^c = B - S$

and $\exists\, E \in \mathcal{Q} \;:\; S = B \cap E$

$\Rightarrow B - S = B \cap (B \cap E)^c = B \cap (B^c \cup E^c)$

$= (B \cap B^c) \cup (B \cap E^c)$

$B - S = B \cap E^c = B \cap (\Omega - E)$

$E \in \mathcal{Q} \Rightarrow E^c \in \mathcal{Q}$

$\therefore\; B \cap E^c \in (B \cap \mathcal{Q})$ //

$\underline{\underline{I}}$: $B = B \cap B$ and $B \in \mathcal{Q}$

$\therefore\; B \in (B \cap \mathcal{Q})$

$\underline{\underline{U}}$: $\{A_i\}$: countable collection in $B \cap \mathcal{Q}$

$\Rightarrow \cup_i A_i = \cup_i (E_i \cap B)$

where $A_i = E_i \cap B \quad \forall\, i \in I$

$\Rightarrow \cup_i A_i = B \cap (\cup_i E_i)$

$\therefore\; (\cup_i A_i) \in B \cap \mathcal{Q}$ //

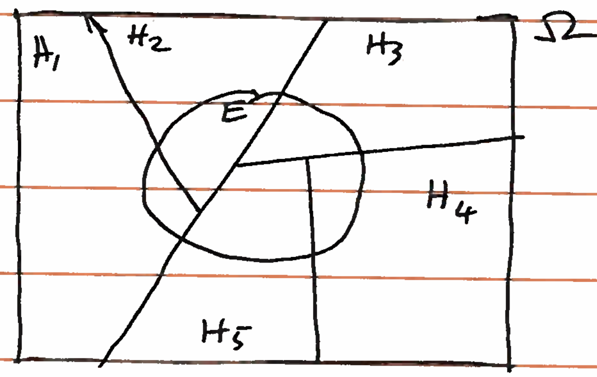
CUT CAT $\therefore$ P-space //

Define: A partition $\{H_k\}_k$ of a space Ω is a collection of sets $\{H_k\}$ such that:

i) $\bigcup_k H_k = \Omega$

ii) $H_k \cap H_j = \phi \;\; \forall\, k \neq j$

e.g.:



Assume measurable sets for partition sets $\{H_k\}$

Qu: How do we express $P(E)$ in terms of the sub-events $\{E \cap H_k\}_k$?

Ans:

Theorem of Total Probability

$$\boxed{P(E) = \sum_k P(H_k) \cdot P(E \mid H_k)}$$

deg(E occurs) = $\sum$ deg(H_k occurs) × deg(H_k causes E)

Proof :

$$E = E \cap \Omega = E \cap \left(\bigcup_k H_k\right) = \bigcup_k (E \cap H_k)$$

The sub-events $\{E \cap H_k\}_k$ are disjoint since $\{H_k\}_k$ is a partition.

$$\Rightarrow \quad P\left(\bigcup_k (E \cap H_k)\right) = \sum_k P(E \cap H_k).$$

$$P(E \cap H_k) = P(E|H_k) \cdot P(H_k)$$

$$\therefore \quad P(E) = \sum_k P(H_k) \cdot P(E|H_k)$$

QED

Qu: Can we express $P(H_j|E)$ in terms of $\{P(H_k)\}_k$ and $\{P(E|H_k)\}_k$? [reverse the conditional]

Ans <u>Bayes Theorem</u>

$$\boxed{P(H_j|E) = \frac{P(H_j) \cdot P(E|H_j)}{\sum_k P(H_k) \cdot P(E|H_k)}}$$

$P(H_j|E)$ = Posterior $\quad P(E|H_k)$: likelihood.

$P(H_j)$: Prior

Proof:

$$P(H_i|E) \overset{P(\cdot|\cdot)}{=\!=} \frac{P(E\cap H_i)}{P(E)}$$

$$\overset{P(\cdot|\cdot)}{=\!=} \frac{P(H_i)\,P(E|H_i)}{P(E)}$$

$$\overset{T.P.}{=\!=} \frac{P(H_i)\cdot P(E|H_i)}{\sum_k P(H_k)\,P(E|H_k)}$$

//

Party Unconditionally To Conquer Bayes

Partition? — Unconditional Probability? $\Rightarrow$ Total Probab.

Partition? — Conditional probability? $\Rightarrow$ Bayes Theorem

Multiplication Theorem

$$\boxed{P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\cdot P(A_2|A_1)\cdot P(A_3|A_1\cap A_2)\cdots P(A_n|A_1\cap\cdots\cap A_{n-1})}$$

Qu How do we prove this statement $\forall n\in\mathbb{Z}^+$?

Ans: Induction

Principle of Mathematical Induction:

Suppose we have statements $\{P_n\}_{n \geq 1}$ and

i) P_1 is true [Basis step]

ii) $P_k \Rightarrow P_{k+1}$ $\forall\, k \geq 1$ [Induction step]

$\therefore$ P_n is true $\forall\, n \geq 1$

notes: ① may start from P_m instead of P_1 ② equivalent to Well-ordering Principle

EX Claim $\sum_{i=1}^{n} i = \frac{1}{2}\, n \cdot (n+1)$ [Gauss' Formula]

Proof: [by induction]

i) P_1: $\sum_{i=1}^{1} i = \frac{1}{2}(1)(1+1) = \frac{2}{2} = 1$

$\therefore$ P_1 is true [Basis step]

ii) Show $P_k \Rightarrow P_{k+1}$ [induction step]

Assume P_k is true [Induction Hypothesis]

i.e. $\sum_{i=1}^{k} i = \frac{1}{2}\, k(k+1)$

Show P_{k+1} is also true

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1) \overset{P_k}{=} \frac{1}{2}\, k(k+1) + (k+1)$$

$$= \frac{1}{2}\left(k^2+3k+2\right) = \frac{1}{2}(k+1)(k+2)$$

$\therefore$ P_{k+1} is true

$\therefore$ P_n is true for $n \geq 1$

Ex [Boole's Inequality]

Claim:

$$\boxed{P\left(\bigcup_{m=1}^{n} A_m\right) \leq \sum_{m=1}^{n} P(A_m)}$$

Proof:

i) [Basis step]

P_1 : $P(A_1) \leq P(A_1)$ is trivially true

P_2 : $P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$

$\because P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

and $P(A_1 \cap A_2) \geq 0$ //

$\therefore$ P_2 is true //

ii) [Induction step] Show $P_k \Rightarrow P_{k+1}$

Assume P_k : $P\left(\bigcup_{m=1}^{k} A_m\right) \leq \sum_{m=1}^{k} P(A_m)$

$$\bigcup_{m=1}^{k+1} A_m = \left(\bigcup^{k} A_m\right) \cup A_{k+1}$$

$$\therefore P\left(\bigcup^{k+1} A_m\right) \overset{P_2}{\leq} P(A_{k+1}) + P\left(\bigcup_{m}^{k} A_m\right)$$

$$\overset{P_k}{\leq} P(A_{k+1}) + \sum_{m=1}^{k} P(A_m)$$

$$\therefore P\left(\bigcup^{k+1} A_m\right) \leq \sum_{m=1}^{k+1} P(A_m)$$

$\therefore$ P_{k+1} is true

$\Rightarrow$ P_n true $\forall\, n \geq 1$

Proof of Multiplication Theorem:

Claim:

$$P\left(\bigcap_{m=1}^{n} A_m\right) = P(A_1)\cdot P(A_2|A_1)\cdots P(A_n|A_1\cap\cdots A_{n-1})$$

Proof:

P_1 trivially true

[Basis step] P_2 : $P(A_1 \cap A_2) = P(A_1)\cdot P(A_2|A_1)$

by defn of conditional probability

[Induction step]: Show $P_k \Rightarrow P_{k+1}$

Assume P_k : $P\left(\bigcap_{m=1}^{k} A_m\right) = P(A_1)\cdot P(A_2|A_1)\cdots P(A_k|A_1\cap\cdots A_{k-1})$

Show P_{k+1}

$$P\left(\bigcap_{m=1}^{k+1} A_m\right) = P\left(\bigcap_{m=1}^{k} A_m \cap A_{k+1}\right)$$

$$\overset{P_2}{=} P\left(\bigcap_{m=1}^{k} A_m\right)\cdot P\left(A_{k+1}\,\middle|\,\bigcap_{m=1}^{k} A_m\right)$$

$$\overset{P_k}{=} P(A_1)\cdot P(A_2|A_1)\cdots P(A_k|A_1\cap\cdots\cap A_{k-1})\cdot P\left(A_{k+1}\,\middle|\,\bigcap_{m=1}^{k} A_m\right)$$

QED

Probability • Limits:

Countable unions of events are measurable. C and U in CUT imply the same is true for countable intersections.

Suppose we have a nested sequence of sets $\{A_k\}_{k=1}^{\infty}$: $A_1 \subset A_2 \subset \dots$

then we can define

$$\boxed{\lim_{k\to\infty} A_k = \bigcup_{k=1}^{\infty} A_k}$$

similarly if we have $\{A_k\}_{k=1}^{\infty}$: $A_1 \supset A_2 \supset \cdots$

then

$$\boxed{\lim_{k\to\infty} A_k = \bigcap_{k=1}^{\infty} A_k}$$

The subsethood relation specify a "linear (or total) (Unique) ordering" on the collection $\{A_k\}$. Limits require this. Hasse Diag. for $\{x,y,z\}$

What can we say about the Probability of such limits?

Theorem: [Continuity of Probability]

i) $P(\lim_{k\to\infty} A_k) = \lim_{k\to\infty} P(A_k)$

if ii) $A_k \subset A_{k+1}$ $\forall k$ $[\lim_{k\to\infty} A_k = \bigcup_{k=1}^{\infty} A_k]$

iii) $A_k \supset A_{k+1}$ $\forall k$ $[\lim_{k\to\infty} A_k = \bigcap_{k=1}^{\infty} A_k]$

Proof of case i/: $A_k \subset A_{k+1} \ \forall k$

Strategy: create a disjoint set sequence $\{B_k\}$ such that

$$\bigcup_k B_k = \bigcup_k A_k$$

Define:

$$B_1 = A_1, \quad B_2 = A_2 - A_1, \quad \cdots, \quad B_k = A_k - A_{k-1}$$

$$\Rightarrow \left\{ \begin{array}{l} \text{ⓐ } \bigcup_k B_k = \bigcup_k A_k \quad \text{ⓑ } B_k \cap B_j = \phi \quad \forall k \neq j \\ \text{ⓒ } P(B_k) = P(A_k) - P(A_{k-1}) \end{array} \right\}$$

call $\lim_{k\to\infty} A_k = A = \bigcup_{k=1}^{\infty} A_k$

$$\therefore P(A) = P\left(\bigcup_k^{\infty} B_k\right) = \sum_{k=1}^{\infty} P(B_k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k) \qquad (*)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} B_k\right)$$

$$\therefore P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n) \quad //$$

case ii/: $A = \lim_{k\to\infty} A_k, \quad A_k \supset A_{k+1} \ \forall k \Rightarrow A_k^c \subset A_{k+1}^c \ \forall k$

$$P(A) = P\left(\bigcap_{k=1}^{\infty} A_k\right)$$

$$= P\left(\left(\bigcup_k^{\infty} A_k^c\right)^c\right) \qquad [\text{De Morgan's}]$$

$$= 1 - P\left(\bigcup_k^{\infty} A_k^c\right)$$

$$= 1 - \lim_{n\to\infty} P(A_n^c) \qquad [\text{by i/}]$$

$$= 1 - \lim_{n\to\infty} [1 - P(A_n)] \qquad [P(A^c) = 1 - P(A)]$$

$$= \lim_{n\to\infty} P(A_n) \quad //$$

Why "Continuity of Probability"?

— Compare to continuity of function f:

f continuous at $x = x_0$

$$\iff f\left(\lim_{x \to x_0} x\right) = \lim_{x \to x_0} f(x)$$

vs Continuity of P

$$P\left(\lim_k A_k\right) = \lim_k P(A_k)$$

Working with linearly ordered sets is rare. But they occur in limsups and liminfs ...

Most sets are: not comparable e.g $\{x, y\} \not\subseteq \{y, z\}$

e.g $A = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \liminf_{n \to \infty} A_n$

$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \limsup_{n \to \infty} A_n$

$B_n = \inf_{k \geq n} A_k$	$B_n = \sup_{k \geq n} A_k$
$= \bigcap_{k \geq n}^{\infty} A_k$	$= \bigcup_{k \geq n}^{\infty} A_k$
[find minimal set]	[find maximal set]
$\Rightarrow B_n \subseteq B_{n+1}$	$\Rightarrow B_{n+1} \subseteq B_n$
$\liminf_{n \to \infty} A_n = \lim_{n \to \infty} B_n$	$\limsup_{n \to \infty} A_n = \lim_{n \to \infty} B_n$

$\therefore \quad A = \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} B_n$

where $B_n \supseteq B_{n+1} \ \forall n \geq 1$ $[A_n \downarrow]$

$\omega \in A \Rightarrow \forall n \geq 1 \ \exists k \geq n : \omega \in A_k$
[in infinitely many A_n]

i.e. outcomes $\omega \in A$ occur infinitely often (i.o.)

So $\limsup_{n\to\infty} A_n$ is often called $\{A_n \text{ i.o.}\}$

Theorem: [Borel-Cantelli Lemma I]

Events : $\{A_n\}_{n\geq 1}$

$$\left[\sum_n P(A_n) < \infty\right] \Rightarrow \left[P\left(\limsup_{n\to\infty} A_n\right) = 0\right]$$

proof:

Define $B_n = \bigcup_{k=n}^{\infty} A_k \Rightarrow B_n \supseteq B_{n+1}, \ B_n \downarrow$

$$\begin{aligned}
\Rightarrow \quad P\left(\limsup_{n\to\infty} A_n\right) &= P\left(\bigcap_{n=1}^{\infty} B_n\right) \\
&= P\left(\lim_{n\to\infty} B_n\right) && [\text{set limit for monotone seq.}] \\
&= \lim_{n\to\infty} P(B_n) && [P\text{-cont}] \\
&= \lim_{n\to\infty} P\left(\bigcup_{k=n}^{\infty} A_k\right) && [\text{def: } B_n] \\
&\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(A_k) && [\text{subadditivity or Boole ineq.}] \\
&= 0
\end{aligned}$$

$\therefore \quad P\left(\limsup_{n\to\infty} A_n\right) = 0$ //

B-C II : $\left\{\{A_n\} \text{ indep and } \sum_{n=1}^{\infty} P(A_n) = \infty\right\} \Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = 1$