This Week: Dr. Osonde Osoba

- Combinatorics
  - Cartesian Products of sets/spaces
  - Sampling with/without Replacement.
    - ${}^nP_k$ / ${}^nC_k$ : ordered vs unordered.
  - Pascal's Formula/Triangle.
- Binomial Theorem
  - Extensions to Multinomial and fractional powers
  - The Binomial pdf: $(\Omega, \mathcal{Q}, P)$ case study.
  - Multinomial pdf

- Series and Limits
  - Absolute vs Conditional convergence
  - Tests for Convergence + ROCs.

H/W 2:

L-G: 2.30, 2.73–2.76

G: 1.43, 1.45, 1.46, 1.49, 1.53, 1.54, 1.60

## I. Cartesian Products:

$$A \times B = \{(a,b) : a \in A,\ b \in B\}$$

e.g. $\mathbb{R} \times \mathbb{R}$, $\mathbb{R}^n$, $[0,1] \times [0,1]$

$\{H,T\} \times \{H,T\}$, $\Omega^n$, $\mathbb{N} \times \mathbb{R}$

Also: function's graph

$$f : X \longrightarrow Y$$

$$G = \{(x, f(x)) : x \in X\} \subset X \times Y$$

An alternative definition of a function:

A function $f : X \rightarrow Y$ is a subset of $X \times Y$ given by its graph $G = \{(x, f(x)) : x \in X\}$

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Qu: # of elements in $A \times B$ ?

Ans: $|A \times B| = |A| \times |B|$ (assuming discrete finite spaces)

i.e. total number of elements (Unique) is $|A| \times |B|$

If each element $\omega \in A \times B$ is equally likely to "occur", we can define the point mass P-measure:

$$P : 2^{A \times B} \longrightarrow [0,1]$$

$$E \subset A \times B$$

$$P(E) = \frac{|E|}{|A \times B|} \quad [\leftarrow \text{Verify CAT}]$$

i.e $P(\{\omega\}) = \dfrac{1}{|A \times B|}$ [Counting measure]

Extend this measure to the case where we sample from the space $A$. $k$ times under the condition that we replace the sampled element each time.

[sampling with replacement]

The effective sample space $\Omega$ is

$$\Omega = A \times \cdots \times A = A^k .$$

Total # of possible sample sequences $= |\Omega| = |A|^k$

Suppose each sampling reduces the space, $A$, by the sampled element. The effective sample space changes.

[sampling without replacement]

Effective sample space $\Omega_p$ :

$$\Omega_p = A \times A - \{\omega_1\} \times A - \{\omega_1, \omega_2\} \times \cdots \times A - \{\omega_1, \cdots, \omega_{k-1}\}$$

$$|\Omega_p| = |A| \times (|A| - 1) \times \cdots \times (|A| - k + 1)$$

let $n = |A|$

$$\Rightarrow |\Omega_p| = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$$

$$= \frac{n}{(n-k)!}$$

$$|\Omega_p| \triangleq {}^nP_k$$

${}^nP_k$ is the # of distinct sequences of length $k$ from a pool of $n$ elements.

e.g. $A = \{0, 0, 1, 1\} \Rightarrow n = 4$

$k = 3$. Sample outcomes from $\Omega_p$ include:

$\vec{\omega}_1 = 010$ $\quad\quad$ $\vec{\omega}_3 = 101$

$\vec{\omega}_2 = 110$ $\quad\quad$ $\vec{\omega}_4 = 001$ ...

Note that $\vec{\omega}_1$ is a rearrangement of $\vec{\omega}_4$. And $\omega_2$ is a rearrangement of $\vec{\omega}_4$. But $\vec{\omega}_1 - \vec{\omega}_4$ are distinct elements in this sample space.

## II Permutations:

Intuitively: A permutation is an action that rearranges the elements of a set.

More rigorously: A permutation is any bijection from a set, $S$, back into $S$. i.e any self-bijection:

$$f : S \longrightarrow S$$

$f$ is bijective [1-1 and onto]

The set of permutation functions on $k$-length sets is the permutation group $S_k$.

The permutation group $S_k$ has $k!$ elements. i.e. there are $k!$ possible ways of rearranging sets with $k$ elements.

We are often not interested in the ordered sequence of outcomes from the sampling experiment. So we count all permutations of a sequence ... as a single unique outcome

This is like defining an equivalence relation "$\sim$":

$\vec{\omega}_a \sim \vec{\omega}_b$ if $\exists\, \sigma \in S_k$ such that $\sigma(\vec{\omega}_a) = \vec{\omega}_b$

[unordered sampling without replacement]:

We start ~~with~~ the ~~samp~~ sample space $\Omega_p$.
But reduce the sample space by lumping together elements that are permutations of each other. Each element has $k!$ permutations. So our new sample space $\Omega_c$ is $k!$ "smaller" in size than $\Omega_p$.

$$|\Omega_c| = \frac{n!}{(n-k)!\,k!} \quad \left[= |\Omega_p| / k!\right]$$

$$|\Omega_c| \triangleq {}^nC_k \quad \left[\text{or } \binom{n}{k}\right]$$

$\binom{n}{k}$ or ${}^{n}C_k$ is the number of combinations or unordered sets of length $k$ from a pool of $n$ elements.

[motivate hypgeom.]

<u>Pascal's Triangle</u> gives $\binom{n}{k}$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | 1 | | | | | | $n=0$ |
| | | | | 1 | | 1 | | | | | $n=1$ |
| | | | 1 | | 2 | | 1 | | | | $n=2$ |
| | | 1 | | 3 | | 3 | | 1 | | | $n=3$ |
| | 1 | | 4 | | 6 | | 4 | | 1 | | $n=4$ |
| 1 | | 5 | | 10 | | 10 | | 5 | | 1 | $n=5$ |

$\longrightarrow$ $k$ from $0$ to $n$

Pascal's triangle depends on <u>Pascal's Formula</u>:

$$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$

$\binom{n}{k}$ and Pascal's Triangle feature heavily in the Binomial Theorem:

$$\boxed{(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}}$$

$$\forall n \in \mathbb{Z}^+, \quad p, q \in \mathbb{R}$$

"Justification":

$$(p+q)^n = \underset{1}{(p+q)} \cdot \underset{2}{(p+q)} \cdots \underset{n}{(p+q)}$$

We can think of $(p+q)^n$ as a sequence summation

$$(p+q)^n = \sum_{j=1}^{2^n} a_j$$

where each $a_j$ are uncombined summands in the expansion of $(p+q)^n$

$$a_j = (a_{j,1}, \cdots a_{j,n})$$

$$a_{j,i} \in \{p, q\}, \quad a_j \in \{p, q\}^n$$

e.g $n = 4$ : $a_j = pqpq, ppqq, pppp, qqqq,$ etc

$\Rightarrow 2^n$ possible $a_j$

Multiplication is commutative $[p \cdot q = q \cdot p]$

$\therefore$ we can combine $a_j$ by ignoring sequence order.

$\binom{n}{k}$ = # ways to fill $a_j$'s $n$ slots with $k$ $p$'s and $(n-k)$ $q$'s

Combining permuted $a_j$'s leaves $(n+1)$ distinct combos

$$\Rightarrow (p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

## Proof of Binomial Theorem : [By Induction]

**Basis step :**

$$n = 1, \quad \binom{1}{0} = \binom{1}{1} = 1$$

$$\Rightarrow \quad (p+q)^n = (p+q)^1 = p + q = \binom{1}{0} p^0 q^1 + \binom{1}{1} p^1 q^0 \checkmark$$

**Induction step :**

Assume $$(p+q)^m = \sum_{k=0}^{m} \binom{m}{k} p^k q^{m-k} \quad (*)$$

$$, \quad (p+q)^{m+1} = (p+q)^m \cdot (p+q)$$

$$\stackrel{(*)}{=} (p+q) \sum_{k=0}^{m} \binom{m}{k} p^k q^{m-k}$$

$$= \underbrace{\left[ \sum_{k=0}^{m} p^{k+1} q^{m-k} \binom{m}{k} \right]}_{(a)} + \underbrace{\left[ \sum_{k=0}^{m} p^k q^{m-k+1} \binom{m}{k} \right]}_{(b)}$$

set $j = k+1$ in (a) and $j = k$ in (b)

$$\Rightarrow (p+q)^{m+1} = \sum_{j=1}^{m+1} p^j q^{m-(j-1)} \binom{m}{j-1}$$

$$+ \sum_{j=0}^{m} p^j q^{m-j+1} \binom{m}{j}$$

$$= \sum_{j=1}^{m+1} p^j q^{m-j+1} \binom{m}{j-1} + \sum_{j=0}^{m} p^j q^{m-j+1} \binom{m}{j}$$

$$\Rightarrow (p+q)^{m+1} = \sum_{j=1}^{m} p^j q^{(m+1)-j} \left[ \binom{m}{j-1} + \binom{m}{j} \right] + p^{m+1} \binom{m}{m} + q^{m+1} \binom{m}{0}$$

Note : i/ $\binom{m}{m} = \binom{m}{0} = 1 = \binom{m+1}{m+1} = \binom{m+1}{0}$ ; ii/ $\binom{m}{j-1} + \binom{m}{j} \overset{\text{Pascal's Formula}}{=\!=} \binom{m+1}{j}$

$$\Rightarrow (p+q)^{m+1} \underset{=}{\text{i/, ii/}} p^{m+1} + q^{m+1} + \sum_{j=1}^{m} \binom{m+1}{j} p^j q^{(m+1)-j}$$

$$\therefore (p+q)^{m+1} \underset{=}{\text{i/}} \sum_{j=0}^{m+1} \binom{m+1}{j} p^j q^{(m+1)-j}$$

$$\therefore (p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}, \quad \forall n \in \mathbb{Z}^+$$

QED

Extensions:

a/ let $n \in \mathbb{Q}$ [e.g $n = a/b$, $a,b \in \mathbb{R}$]

then $$(p+q)^n = \sum_{k=0}^{\infty} \frac{(n)_k}{k!} p^k q^{n-k}$$

where $(n)_k = n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)$

b/ Define $$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!}$$

where $\sum_{i=1}^{m} k_i = n$

Observe that $$\binom{n}{k} = \binom{n}{k, (n-k)}$$

Multinomial Theorem:

$$\boxed{(x_1 + \cdots + x_m)^n = \sum_{k_1, \ldots, k_m}^{n} \binom{n}{k_1, \ldots, k_m} x_1^{k_1} \cdots x_m^{k_m}}$$

**Ex:** 10 coin flips: $P(H) = p = 1-q$, $q = P(T)$

(a) $P(7 \text{ heads}) = ?$

$$= \binom{10}{7} p^7 q^{10-7}$$

(b) $P(8T \mid 1^{st}\ 5 \text{ flips are } T) = ?$

$$= P(3T \text{ out of } 5 \text{ flips})$$

$$= \binom{5}{2} p^2 q^3$$

**Ex:** Prove/Disprove:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Proof: $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$

**Ex:** Single Die rolled 10 times

$P(3 \text{ six's}, 4 \text{ two's}) = ?$

$P(\text{six}) = 1/6 \triangleq p_1$ $\quad P(\text{two}) = 1/6 \triangleq p_2$

$P(\text{not six or two}) = p_3 = 1 - p_1 - p_2 = 4/6$

$P(3 \text{ six's}, 4 \text{ two's}) = P(3 \text{ six's}, 4 \text{ two's}, 3 \text{ else})$

$$= \binom{10}{3, 4, 3} p_1^3 p_2^4 p_3^3$$

**Ex** [Parity Check]

Given bit-packets of length $n$: $(n-1)$ data bits and 1 parity check bit. Check-bit set as follows

$$\text{Check-bit} = \begin{cases} 1 & \text{if \# of "on" bits in data packet odd} \\ 0 & \text{if \# of "off" bits in data packet even} \end{cases}$$

So a bad bit-packet has odd number of 1's. What is the probability of observing a bad bit-packet is the probability of a bit-flip is $p$ for every bit in the packet?

**Ans:** $P(\text{bad Packet}) = P(\text{\# of 1's is odd})$

$$= P(\text{odd \# of bit-flips})$$

$$\therefore P(\text{bad Packet}) = \sum_{\substack{k \geq 1 \\ k \text{ odd}}}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

## II// Limits and Series:

A simple way to define P-measures on a countable space is to use the point-mass measure:

$$\forall\ \omega_k \in \Omega \qquad P(\{\omega_k\}) = a_k \qquad [\text{let } \Omega = \{\omega_k\}_{k \in \mathbb{N}}]$$

CAT constraints require that

$$\sum_{k=1}^{\infty} P(\{\omega_k\}) = 1 \qquad [\because \bigcup_k \{\omega_k\} = \Omega]$$

$$\text{i.e} \quad \sum_{k=1}^{\infty} a_k = 1$$

This gives a way for identifying measures with summable sequences $\{a_k\}_{k \in \mathbb{N}}$.

### Definition:

(a) $$\lim_{k \to \infty} a_k = L$$

$$\iff \forall \varepsilon > 0 \quad \exists n_0 \in \mathbb{N} : \forall n \geq n_0$$
$$|a_n - L| < \varepsilon$$

(b) $$\sum_{k=1}^{\infty} a_k \triangleq \lim_{n \to \infty} \sum_{k=1}^{n} a_k$$

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$$\lim_{k \to \infty} a_k = \infty \quad \not\Leftarrow\!\!\!\Rightarrow \quad \lim_{k \to \infty} a_k \text{ does not exist}$$

e.g $\{a_k\}_{k \geq 1}$ ; $a_k = 1/k$

claim: $\lim a_k = 0$

pick any $\varepsilon > 0$ $\Rightarrow n_0 = \max(1, 1 + \lceil 1/\varepsilon \rceil)$

say $\varepsilon = 0.1$ $\Rightarrow n_0 = 1 + 10 = 11$

$a_{11} = 1/11 = 0.0909$, $a_{20} = 1/20 = 0.05$

$\Rightarrow |a_{11} - 0| < \varepsilon$, $|a_{20} - 0| < \varepsilon$.

We are more interested in summable series.

Define the sum sequence (or series)

$$S_n = \sum_{k=1}^{n} a_k$$

$\boxed{\text{Qu: is } |S_n| < \infty \text{ always?}}$

and $S = \lim_{n \to \infty} S_n$

e.g $a_k = r^k$

$$\Rightarrow S_n = \sum_{k=1}^{n} r^k$$

$$\Rightarrow S_n = r + r^2 + \dots + r^n$$

$$-(rS_n = r^2 + r^3 + \dots + r^{n+1})$$

$$(1-r)S_n = r + r^{n+1}$$

$$\therefore \boxed{S_n = \frac{r - r^{n+1}}{1-r}}$$

Similarly prove $\sum_{k=0}^{n} r^k = \dfrac{1-r^{n+1}}{1-r}$

$$\lim_{n\to\infty} S_n = \frac{r}{1-r} - \cancel{\frac{\lim_{n\to\infty} r^{n+1}}{1-r}} \nearrow \begin{cases} \infty & \text{if } |r| \geq 1 \\ 0 & \text{if } |r| < 1 \end{cases}$$

$\Rightarrow \lim_{n\to\infty} S_n = \dfrac{r}{1-r}$ when $|r| < 1$

$$\boxed{S = \frac{r}{1-r}} \quad \text{when } |r| < 1$$

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$\sum_{n=0}^{\infty} a_n$ converges absolutely

iff $\sum_{n=0}^{\infty} |a_n|$ converges

$\sum_{n=0}^{\infty} a_n$ converges

iff $\lim_{n\to\infty} S_n = S$ converges

$\sum_{n=0}^{\infty} a_n$ converges conditionally

iff $\begin{cases} \sum_{n=0}^{\infty} a_n & \text{converges} \\ \text{and} & \\ \sum_{n=0}^{\infty} a_n & \text{diverges} \end{cases}$

e.g. ~~of~~ [sequences]

$1/n!$, $1/n^p$, $r^n/n!$, $3^n/n^2$, $n^n/n!$, $(-1)^{n+1}/\sqrt{n}$

$r^n$, $(-1)^n$, ... [(sub)/(super) polynomial sequences]

Qu: How do we test the (absolute) convergence of a given series summation?

Ans: Convergence tests include:

i/ - Ratio Test

ii/ - P-series Test

iii/ - Alternating Series Test

Also:
- Integral test, - Direct comparison test
- root test

i/ Ratio Test:

Test $\sum_{n=0}^{\infty} a_n$ for convergence

let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

Then

- $L < 1 \Rightarrow$ Absolute Convergence
- $L > 1 \Rightarrow$ Divergence
- $L = 1 \Rightarrow$ Test fails (!!)

ii/ <u>P-series Test</u> :

Test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for convergence

– converges if $p > 1$

– diverges if $p \leq 1$

iii/ <u>Alternating Series Test</u> :

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

if $\begin{cases} a_k \geq a_{k+1} > 0 & \forall k \in \mathbb{Z}^+ \\ \lim_{n\to\infty} a_n = 0 \end{cases}$

<u>Ex</u> :

(a) $S = \sum_{n=1}^{\infty} 3^n/n^2$

$a_n = 3^n/n^2$ ; $L = \lim_n \left| \frac{3^{n+1} \cdot n^2}{(n+1)^2 \cdot 3^n} \right| = \lim_n \frac{3 \cdot n^2}{(n+1)^2}$

$\lim_{n\to\infty} \frac{3\ n^2}{(n+1)^2} = 3 \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 3 > 1$

$\therefore$ S diverges.

[exponential $(3^n)$ "drowns" polynomial $(n^2)$]

↳ true for any polynomial $n^p$

(b) $S = \sum_{n=0}^{\infty} r^n$

$L = \lim_n \left| r^{n+1} / r^n \right| = \lim_n |r| = |r|$

$\therefore$ $S$ converges absolutely if $|r| < 1$

(c) $S = \sum_{n=0}^{\infty} r^n / n!$

$L = \lim_n \left| \frac{r^{n+1}}{r^n} \cdot \frac{n!}{(n+1)!} \right| = \lim_n \left| r \cdot \frac{1}{n+1} \right|$

$= |r| \cdot \lim_{n \to \infty} \left| \frac{1}{n+1} \right| = 0$

$\therefore$ $S$ converges absolutely

[ factorial "drowns" exponential ]

<u>Remedial</u> <u>Questions</u> :

(a) $\lim_n (a_n + b_n) = ?$

(b) $\lim_n (a_n \cdot b_n) = ?$

(c) $\lim_n (a_n / b_n) = ?$

(d) $\lim_n (a) = ?$

(e) Describe the sets $\{x \mid |x| < 5\}$ and $\{x \mid |x| > 5\}$

(f) $[a_n \in X, \forall n] \overset{??}{\Longrightarrow} [\lim_n a_n \in X]$

<u>Ex</u> (d) $S = \sum_{n=1}^{\infty} (-1)^n / \sqrt{n}$ [alternating series]

$a_n = 1/\sqrt{n} \geq a_{n+1} = 1/\sqrt{n+1}$

$\Rightarrow$ $S$ converges $[S = \ln 2]$

But: $\left| (-1)^n / \sqrt{n} \right| = \frac{1}{n^p}$; $p = 0.5 \leq 1$

$\Rightarrow \sum_{n=1}^{\infty} \left| (-1)^n / \sqrt{n} \right|$ diverges

$\therefore$ The series is <u>conditionally</u> <u>convergent</u>

## Riemann's Rearrangement Theorem:

$\left[\sum_{n=0}^{\infty} a_n \text{ conditionally convergent}\right]$

$\Rightarrow \left[\forall x \in \mathbb{R} \ \exists \text{ a rearrangement } \sum_n^{\infty} b_n \text{ of } \sum_n^{\infty} a_n \ \ni \ \sum_{n=0}^{\infty} b_n = x\right]$

This means that infinite sequences with (+ve) conditionally convergent sums cannot be used for P-measures;

They will not give countably additive (CA) measures

## Power series: $\left[\sum_{n=0}^{\infty} a_n x^n\right]$

Power series are expansions of functions in terms of ~~#~~ finite or infinite summations of polynomials.

e.g.

MacLaurin series expansion for $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n$$

Taylor series for $f(x)$ near $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n$$

Laurent series for $f(x)$

$$f(x) = \sum_{n=0}^{\infty} \frac{b_n}{(x-x_0)^n}$$

Series expansions are generally valid for open subsets smaller than the entire domain $x \in X$. These open subsets (regions of convergence, ROC) are the sets over which the power series converges.

e.g. Find the ROC for

$$\sum_{n=1}^{\infty} \frac{x^n}{n+4}$$

Use Ratio test

$$L = \lim_{n} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n+4}{n+5} \right| = \lim_{n \to \infty} |x| \cdot \left| \frac{n}{n+5} + \frac{4}{n+5} \right|$$

$L = |x|$ $\therefore$ series converges if $\boxed{|x| < 1}$

@ $x = -1$ series converges by Alternating series test

@ $x = 1$ series diverges by Integral test

$\hookrightarrow \int_1^{\infty} \frac{dx}{x+4} \overset{?}{<} \infty$

$\therefore$ ROC $= [-1, 1)$

$$\sum_{n=1}^{\infty} \frac{(-3)^n x^{n+1}}{n}$$

Other power series expansions:

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (x+1)^n \qquad x \in \mathbb{R}$$

$$f(x) = \sum_{n=0}^{\infty} x^n = (1-x)^{-1} \qquad |x| < 1$$

$$f(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)! = \cosh x \qquad |x| \in \mathbb{R}, \mathbb{C}$$

Exp(x):

$$\boxed{\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

valid $\forall\, x \in \mathbb{R}$ or $\mathbb{C}$

Also: $e^x = \lim_{n\to\infty} (1 + x/n)^n$

exp(x) is
- super-polynomial [Gamma fxn]
- sub-factorial [Stirling's Approximation]

i.e $\lim_{n\to\infty} n^p \cdot e^{-n} = 0 \quad \forall\, p \geq 1$

and $\lim_{n\to\infty} (n!) \cdot e^{-n} = \infty$

Gamma function:

$$\boxed{\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \cdot e^{-x}\, dx} \qquad \alpha \in \mathbb{R}^+$$

$$\Gamma(\alpha+1) = \alpha\, \Gamma(\alpha), \qquad n \in \mathbb{Z}^+ \Rightarrow \Gamma(n+1) = n!$$

$\Gamma$ is the unique logconvex extension of the factorial to $\mathbb{R}^+$

Stirling's Approximation:

$$\boxed{\ln n! \approx n \ln n - n \approx n \ln n}$$

# Borel-Cantelli Lemma I (again)

$\{A_n\}_{n\geq 1}$

$$\left[\sum_n P(A_n) < \infty\right] \Rightarrow \left[P(\limsup_{n\to\infty} A_n) = 0\right]$$

## Proof ①:

$$A = \limsup_{n\to\infty} A_n \Rightarrow A = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$$

$$B_m \triangleq \bigcup_{n=m}^{\infty} A_n \Rightarrow B_m \supset B_{m+1}$$ [union over fewer sets]

$$\Rightarrow \lim_{m\to\infty} B_m = \bigcap_{m=1}^{\infty} B_m = A$$

$$\Rightarrow P(A) = P(\lim_{m\to\infty} B_m) \stackrel{\text{P.Cont}}{=} \lim_{m\to\infty} P(B_m)$$

$$\stackrel{\text{def } B_m}{=} \lim_m P\left(\bigcup_{n=m}^{\infty} A_n\right)$$

$$\stackrel{\text{Boole}}{\leq} \lim_m \sum_{n=m}^{\infty} P(A_n) = 0$$ [$\because \lim_{m\to\infty} \sum_{n=m}^{\infty} P(A_n) = 0$ if $\sum P(A_n) < \infty$]

$$P(A) = 0$$

## Proof ②:

$$P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n\right) \leq P\left(\bigcup_{n=m}^{\infty} A_n\right)$$ [$\because \bigcap B_m \subseteq B_m \ \forall m$]

$$\leq \sum_{n=m}^{\infty} P(A_n)$$