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This Week:

- Continuous Random Variables
  - Review: Random Variables, Continuity.
  - Cumulative Distribution Functions
  - BEG CUP Distributions
- Bayesian Inference with Random Variables
  - Conjugacy
  - Examples

H/W 4:

| | | | | |
|---|---|---|---|---|
| L-G: | 3.53, | 3.54, | 3.57 – 3.59 | |
| | 4.5, | 4.25, | 4.26, | 4.34 |
| | 4.63, | 4.64, | 4.67, | 4.68 |

I// **Review: Random Variables**

$(\Omega, \mathcal{Q}, P)$ : probability space

$(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ : measurable space

Given a function

$$X : \Omega \longrightarrow \mathbb{R}$$

Then $X$ is a <u>random variable</u>

$$\Longleftrightarrow \quad X^{-1}(A) \in \mathcal{Q} \quad \forall A \in \mathcal{B}(\mathbb{R})$$

i.e <u>P</u>ullbacks $(X^{-1}(A))$ are <u>A</u>lways <u>M</u>easurable <u>[PAM]</u>

(for any Borel-measurable set $A$)

OR $\Longleftrightarrow$ $X : (\Omega, \mathcal{Q}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a <u>measurable map</u>

Define the <u>Distribution function</u> $P_X(\cdot)$

as

$$\boxed{\begin{aligned} P_X(A) &= P(X \in A) = P(X^{-1}(A)) \\ &= P(\{\omega \in \Omega \mid X(\omega) \in A\}) \end{aligned}}$$

where $A$ is any <u>Borel set</u> (i.e. $A \in \mathcal{B}(\mathbb{R})$)

Recall that Borel sets include open sets (e.g. $(a,b)$ $(-\infty, a)$) and any sets that can be written as countable unions and intersections of open sets e.g $\{a\}$ where $a \in \mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{N}$. The measurability of $X$ allows the composition $P \circ X^{-1}(A)$ since $P : \mathcal{Q} \to [0,1]$ and $X^{-1}(A) \in \mathcal{Q}$.

When $X$ is a discrete random variable
(i.e $|X(\Omega)| < |\mathbb{R}|$) [Qu: What does this mean? Why use this definition?]
then we only need to deal with the
distribution function over a discrete domain. So

$$A \in \{\{k\} : k \in X(\Omega)\} \subset \mathcal{B}(\mathbb{R})$$

This gives us the probability mass function (PMF)

$$p_X(k) = P(X^{-1}(\{k\}))$$
$$= P(X = k)$$

giving the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P\circ X^{-1})$
or more precisely $(\mathbb{Z}, \mathbb{Z} \cap \mathcal{B}(\mathbb{R}), P\circ X^{-1})$

When $X$ is a continuous random variable
(i.e $|X(\Omega)| = |\mathbb{R}|$) [Note: this defn is not precise enough to define continuous r.v.s. We'll see why]
then we use the half-open intervals

$$A \in \{(-\infty, x] : x \in \mathbb{R}\} \subset \mathcal{B}(\mathbb{R})$$

This gives the cumulative distribution function (CDF)

$$F_X(x) = P(X \in (-\infty, x])$$
$$= P(X \le x)$$

if $F_X(\cdot)$ is absolutely continuous then $\exists\, f_X(x)$:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt.$$

## Review of Continuity: (regular, uniform, and absolute)

$$F : \mathbb{R} \longrightarrow \mathbb{R}$$

$F$ continuous

$$\Longleftrightarrow \forall x \in \mathbb{R}, \ \forall \varepsilon > 0 \ \exists \delta(x) > 0 :$$
$$\forall z \in \mathbb{R} \quad |x - z| < \delta(x) \Rightarrow |F(x) - F(z)| < \varepsilon$$

$F$ uniformly continuous (U.C.)

$$\Longleftrightarrow \forall \varepsilon > 0, \ \exists \delta > 0 \ \forall x, z \in \mathbb{R} :$$
$$|x - z| < \delta \Rightarrow |F(x) - F(z)| < \varepsilon$$

$F$ absolutely continuous (A.C.)

$\Longleftrightarrow \forall \varepsilon > 0, \ \exists \delta > 0 \ \forall$ finite sequences of disjoint intervals $\{(a_i, b_i)\}_{i=1}^{K}$ :

$$\sum_{i=1}^{K} (b_i - a_i) < \delta \Rightarrow \sum_{i=1}^{K} |F(b_i) - F(a_i)| < \varepsilon$$

absolute continuity $\underset{\nLeftarrow}{\Rightarrow}$ Uniform continuity $\underset{\nLeftarrow}{\Rightarrow}$ continuous

$F$ A.C. (w.r.t. Lebesgue measure)

$\Rightarrow \begin{cases} \text{i/ } F \text{ is differentiable almost-everywhere} \\ \text{ii/ } \exists \text{ a Lebesgue-integrable function, } f(x), \\ \qquad F(x) = F(a) + \int_a^x f(t)\, dt \qquad [\text{FTC}] \end{cases}$

A CDF $F$ is absolutely continuous

$\iff \exists$ a probability density function (PDF) $f$:

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

$\therefore$ a/ $f = dF/dx$

b/ $\int_{-\infty}^{\infty} f(x)\, dx = 1$

CDFs satisfy the following properties in general:

i/ $F$ is non decreasing

ii/ $F$ is right-continuous

iii/ $F$ has left limits $\forall x \in \mathbb{R}$

iv/ $\lim_{x \to \infty} F(x) = 1$

v/ $\lim_{x \to -\infty} F(x) = 0$

- Absolutely continuous CDFs, $F_{ac}$, assign non-zero measure only to non-empty open sets (sets with non-zero Lebesgue measure).

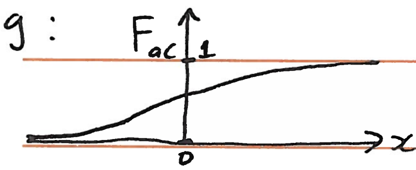
- Discrete CDFs, $F_d$, assign non-zero measure only to singleton sets and countable unions thereof (sets w̄ zero Lebesgue measure)

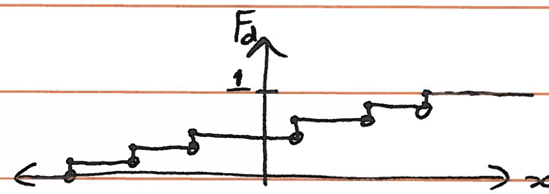
- Singularly continuous CDFs, $F_{sc}$, are continuous-but-not-A.C. CDFs. Hence they have no PDFs. $F_{sc}$ assigns non-zero measure to sets w̄ zero-Lebesgue measure also. But these sets are not (made up of) singleton sets and may even be uncountable.

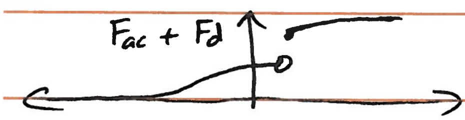
General CDFs are decomposable into these 3 components:

$$F_{Gen} = F_{ac} + F_d + F_{sc}$$

e.g:




$F_{sc}$ : lookup Cantor Function

(Continuous everywhere but differentiable nowhere. Difficult to draw)

## II// Continuous Random Variables:

B $\boxed{E}$ $\boxed{G}$ $\boxed{C}$ $\boxed{U}$ P

(a) (d) (c) (b)

### ⓐ Exponential:

The most general distribution in this class is the gamma distribution from which we get the exponential, Chi-squared, and n-Erlang as special cases. This class of pdfs uses the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot \exp(-x)\, dx \qquad \alpha \in \mathbb{R}^+$$

as a foundation.

$X \sim \gamma(\alpha, \theta)$:

$$f(x) = \frac{x^{\alpha-1} \cdot \exp(-x/\theta)}{\Gamma(\alpha)\ \theta^{\alpha}}$$

$x \in \mathbb{R}^+$, $\alpha \in \mathbb{R}^+$, $\theta \in \mathbb{R}^+$

i/ $\alpha = 1 \Rightarrow X \sim \exp(\theta)$ ; $f(x) = \exp(-x/\theta)/\theta$

ii/ $\alpha \in \mathbb{Z}^+ \Rightarrow X \sim$ Erlang$(\theta)$

iii/ $\underset{r \in \mathbb{Z}^+}{\alpha = r/2}$, $\theta = 2 \Rightarrow X \sim \chi^2(r)$

(b) Uniform :

$X \sim U(a,b)$:

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$

$U$ is a special case of $Beta(\alpha, \beta)$

$X \sim Beta(\alpha, \beta)$:

$$f(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)} \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1} & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$$

$\alpha = \beta = 1 \Rightarrow X \sim U(0,1)$

(c) Cauchy :

$X \sim C(m, d)$

$$f(x) = \frac{1}{\pi \cdot d \cdot \left(1 + \left(\frac{x-m}{d}\right)^2\right)}$$

m : location $\in \mathbb{R}$

d : dispersion $\in \mathbb{R}^+$

Standard Cauchy : $Z \sim C(0,1)$

$$f(z) = \frac{1}{\pi(1 + z^2)}$$

(d) Gaussian :

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$\mu \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}^+$

Standard Normal : $Z \sim N(0,1)$

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$$

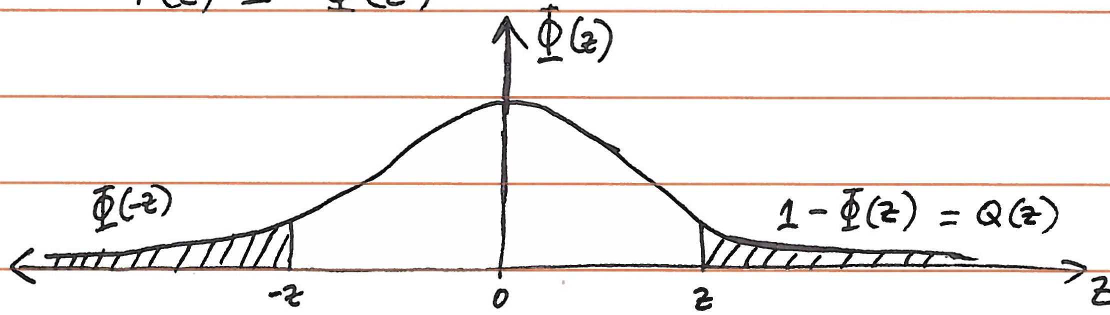
Standardization : $std(X) = \left(\frac{X-\mu}{\sigma}\right)$

## Table of Standard Normal CDF values:

$Z \sim N(0,1)$

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2} \, dw$$

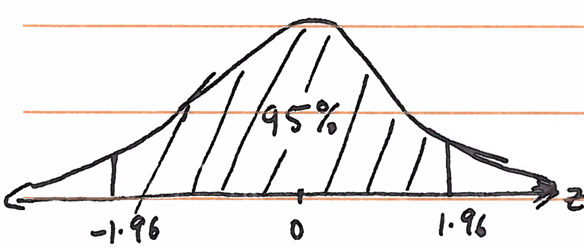
$$F(z) = \Phi(z)$$

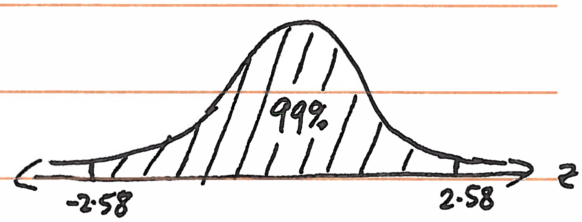



By symmetry $\Rightarrow \Phi(-z) = 1 - \Phi(z)$

$\Phi(z) = P(Z \leq z)$ ; $Q(z) = P(Z > z)$

$$\Phi(z) + Q(z) = 1$$




$P(-1.96 \leq Z \leq 1.96) = 0.95$

$P(-2.58 \leq Z \leq 2.58) = 0.99$

## III// Bayesian Inference for Continuous rvs :

Bayes Theorem for Events :

$$P(H_k|E) = \frac{P(H_k) \cdot P(E|H_k)}{\sum_j P(H_j) \cdot P(E|H_j)}$$

Where $P(H_k|E)$ gives the extent to which the event $E$ supports the occurrence of the event $H_k$.

We can formulate the continuous analogue :

$$\boxed{f(\theta/x) = \frac{h(\theta) \cdot g(x|\theta)}{\int_{-\infty}^{\infty} h(\theta)\, g(x|\theta)\, d\theta}} = \frac{h \cdot g}{Q(x)}$$

$h(\theta)$ : prior pdf

$g(x|\theta)$ : likelihood function

$f(\theta|x)$ : Posterior pdf.

i.e. How does observing $X = x$ modify our $\overset{[h(\theta)]}{\text{prior beliefs}}$ about $\theta$ ?

## Conjugacy :

Bayes theorem maps the pair $(h, g)$ to $f$.

$$f \propto h \cdot g$$

Conjugacy refers to the cases where $h$ and $f$ belong to the same pdf family.

i) e.g $\overset{h}{(\text{Beta}}, \overset{g}{\text{Binomial})} \longrightarrow \overset{f}{\text{Beta}}$

"Beta is conjugate to Binomial"

i.e $X \sim b(n, \theta) \Rightarrow g(x|\theta) = \binom{n}{x} \theta^{x} (1-\theta)^{n-x}$

$\theta \sim \text{Beta}(\alpha, \beta) \Rightarrow h(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)} \cdot \theta^{\alpha-1} (1-\theta)^{\beta-1}$

$$\Rightarrow f(\theta|x) = \frac{\cancel{\binom{n}{x}} \theta^{x} (1-\theta)^{n-x} \cdot \cancel{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)}} \cdot \theta^{\alpha-1} \cdot (1-\theta)^{\beta-1}}{\int_0^1 \cancel{\binom{n}{x}} \cdot \cancel{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \cdot \Gamma(\beta)}} \cdot \theta^{x} \cdot (1-\theta)^{n-x} \cdot \theta^{\alpha-1} \cdot (1-\theta)^{\beta-1} \, d\theta}$$

$$f(\theta|x) = \frac{\theta^{(\alpha+x)-1} \cdot (1-\theta)^{(\beta+n-x)-1}}{\int_0^1 \theta^{x+\alpha-1} \cdot (1-\theta)^{n-x+\beta-1} \, d\theta}$$

By definition:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, d\theta$$

$$\Rightarrow f(\theta|x) = \frac{\theta^{(\alpha+x)-1} \cdot (1-\theta)^{(\beta+n-x)-1}}{B(\alpha+x, \beta+n-x)}$$

i.e $\Theta|X \sim Beta(\alpha+x, \beta+n-x)$

ii) (Gamma, Poisson) $\longrightarrow$ Poisson

$$\Theta \sim \Gamma(\alpha,\beta) \Rightarrow h(\theta) = \frac{\theta^{\alpha-1} \cdot \exp(-\theta/\beta)}{\Gamma(\alpha)\cdot\beta^{\alpha}}$$

$$X|\Theta \sim Poisson(\theta) \Rightarrow g(x|\theta) = \frac{e^{-\theta}\cdot\theta^{x}}{x!}$$

$$\Rightarrow f(\theta|x) = \frac{\theta^{\alpha-1}\cdot\exp(-\theta/\beta)\cdot e^{-\theta}\cdot\theta^{x}}{x!\cdot\Gamma(\alpha)\cdot\beta^{\alpha}} \Big/ \int_0^{\infty} \frac{\theta^{x+\alpha-1}\cdot\exp(-\theta-\theta/\beta)}{x!\cdot\Gamma(\alpha)\cdot\beta^{\alpha}}\, d\theta$$

$$= \frac{\theta^{x+\alpha-1}\cdot\exp(-\theta-\theta/\beta)}{\int_0^{\infty}\theta^{x+\alpha-1}\cdot\exp(-\theta(1+1/\beta))\, d\theta}$$

let $s = \theta\left(\frac{\beta+1}{\beta}\right)$

$$f(\theta|x) = \frac{\theta^{\alpha+x-1}\cdot\exp\left[-\theta/\left(\frac{\beta}{1+\beta}\right)\right]}{\left(\frac{\beta}{1+\beta}\right)^{\alpha+x}\cdot\int_0^{\infty} s^{x+\alpha-1}\cdot e^{-s}\, ds}$$

$$f(\theta|x) = \frac{\theta^{\alpha+x-1}\cdot\exp\left(-\theta/\left(\frac{\beta}{1+\beta}\right)\right)}{\Gamma(\alpha+x)\cdot\left(\frac{\beta}{1+\beta}\right)^{\alpha+x}}$$

i.e $\Theta|X \sim \Gamma\left(\alpha+x, \frac{\beta}{1+\beta}\right)$

Cauchy: $X \sim C(m, d)$

$$f(x) = \frac{1}{\pi d \left(1 + \left(\frac{x-m}{d}\right)^2\right)}$$

## Continuous Distributions

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

**Beta**
$0 < \alpha$
$0 < \beta$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1$$

$$\mu = \frac{\alpha}{\alpha+\beta}, \qquad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

**Chi-square**
$\chi^2(r)$
$r = 1, 2, \ldots$

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \qquad 0 \le x < \infty$$

$$M(t) = \frac{1}{(1-2t)^{r/2}}, \qquad t < \frac{1}{2}$$

$$\mu = r, \qquad \sigma^2 = 2r$$

**Exponential**
$0 < \theta$

$$f(x) = \frac{1}{\theta} e^{-x/\theta}, \qquad 0 \le x < \infty$$

$$M(t) = \frac{1}{1-\theta t}, \qquad t < \frac{1}{\theta}$$

$$\mu = \theta, \qquad \sigma^2 = \theta^2$$

**Gamma**
$0 < \alpha$
$0 < \theta$

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \le x < \infty$$

$$M(t) = \frac{1}{(1-\theta t)^\alpha}, \qquad t < \frac{1}{\theta}$$

$$\mu = \alpha\theta, \qquad \sigma^2 = \alpha\theta^2$$

**Normal**
$N(\mu, \sigma^2)$
$-\infty < \mu < \infty$
$0 < \sigma$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty$$

$$M(t) = e^{\mu t + \sigma^2 t^2/2}$$

$$E(X) = \mu, \qquad \mathrm{Var}(X) = \sigma^2$$

**Uniform**
$U(a, b)$
$-\infty < a < b < \infty$

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b$$

$$M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, \qquad t \neq 0; \qquad M(0) = 1$$

$$\mu = \frac{a+b}{2}, \qquad \sigma^2 = \frac{(b-a)^2}{12}$$

$$\Gamma(\alpha) \equiv \int_0^\infty x^{\alpha-1} e^{-x}\, dx$$

$\therefore \Gamma(1) = 1 \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$\Gamma(n) = (n-1)!$ if $n \in \mathbb{Z}^+$

Multivariate Hypergeometric

$$f(x) = \frac{\binom{N_1}{x_1}\binom{N_2}{x_2}\cdots\binom{N_k}{x_k}}{\binom{N}{n}}$$

$N_1 + \cdots + N_k = N$

$x_1 + \cdots + x_k = n$

# Discrete Distributions

**Bernoulli**
$0 < p < 1$

$f(x) = p^x(1-p)^{1-x}, \quad x = 0, 1$

$M(t) = 1 - p + pe^t$

$\mu = p, \quad \sigma^2 = p(1-p)$

**Binomial**
$b(n, p)$
$0 < p < 1$

$f(x) = \dfrac{n!}{x!(n-x)!} p^x(1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n$

$M(t) = (1 - p + pe^t)^n$

$\mu = np, \quad \sigma^2 = np(1-p)$

**Geometric**
$0 < p < 1$

$f(x) = (1-p)^{x-1}p, \quad x = 1, 2, 3, \ldots$

$M(t) = \dfrac{pe^t}{1-(1-p)e^t}, \quad t < -\ln(1-p)$

$\mu = \dfrac{1}{p}, \quad \sigma^2 = \dfrac{1-p}{p^2}$

$\therefore P(X > k) = q^k$

**Hypergeometric**
$N_1 > 0, \quad N_2 > 0$
$N = N_1 + N_2$

$f(x) = \dfrac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}}, \quad x \le n, x \le N_1, n - x \le N_2$

$\mu = n\left(\dfrac{N_1}{N}\right), \quad \sigma^2 = n\left(\dfrac{N_1}{N}\right)\left(\dfrac{N_2}{N}\right)\left(\dfrac{N-n}{N-1}\right)$

**Negative Binomial**
$0 < p < 1$
$r = 1, 2, 3, \ldots$

$f(x) = \dbinom{x-1}{r-1}p^r(1-p)^{x-r}, \quad x = r, r+1, r+2, \ldots$

$M(t) = \dfrac{(pe^t)^r}{[1-(1-p)e^t]^r}, \quad t < -\ln(1-p)$

$\mu = r\left(\dfrac{1}{p}\right), \quad \sigma^2 = \dfrac{r(1-p)}{p^2}$

**Poisson**
$0 < \lambda$

$f(x) = \dfrac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$

$M(t) = e^{\lambda(e^t - 1)}$

$\mu = \lambda, \quad \sigma^2 = \lambda$

**Uniform**
$m > 0$

$f(x) = \dfrac{1}{m}, \quad x = 1, 2, \ldots, m$

$\mu = \dfrac{m+1}{2}, \quad \sigma^2 = \dfrac{m^2-1}{12}$

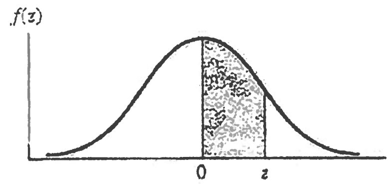
Multinomial

$$f(x) = \frac{n!}{x_1!\,x_2!\cdots x_k!}\,p_1^{x_1}\cdots p_k^{x_k}$$

$x_1 + \cdots + x_k = n$

$p_1 + \cdots + p_k = 1$

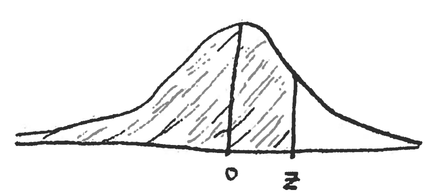
TABLE 4 **Normal Curve** Areas




| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .0000 | .0040 | .0080 | .0120 | .0160 | .0199 | .0239 | .0279 | .0319 | .0359 |
| .1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0753 |
| .2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 |
| .3 | .1179 | .1217 | .1255 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 |
| .4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 |
| .5 | .1915 | .1950 | .1985 | .2019 | .2054 | .2088 | .2123 | .2157 | .2190 | .2224 |
| .6 | .2257 | .2291 | .2324 | .2357 | .2389 | .2422 | .2454 | .2486 | .2517 | .2549 |
| .7 | .2580 | .2611 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 |
| .8 | .2881 | .2910 | .2939 | .2967 | .2995 | .3023 | .3051 | .3078 | .3106 | .3133 |
| .9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 |
| 1.1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3830 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3980 | .3997 | .4015 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4429 | .4441 |
| 1.6 | .4452 | .4463 | .4474 | .4484 | .4495 | .4505 | .4515 | .4525 | .4535 | .4545 |
| 1.7 | .4554 | .4564 | .4573 | .4582 | .4591 | .4599 | .4608 | .4616 | .4625 | .4633 |
| 1.8 | .4641 | .4649 | .4656 | .4664 | .4671 | .4678 | .4686 | .4693 | .4699 | .4706 |
| 1.9 | .4713 | .4719 | .4726 | .4732 | .4738 | .4744 | .4750 | .4756 | .4761 | .4767 |
| 2.0 | .4772 | .4778 | .4783 | .4788 | .4793 | .4798 | .4803 | .4808 | .4812 | .4817 |
| 2.1 | .4821 | .4826 | .4830 | .4834 | .4838 | .4842 | .4846 | .4850 | .4854 | .4857 |
| 2.2 | .4861 | .4864 | .4868 | .4871 | .4875 | .4878 | .4881 | .4884 | .4887 | .4890 |
| 2.3 | .4893 | .4896 | .4898 | .4901 | .4904 | .4906 | .4909 | .4911 | .4913 | .4916 |
| 2.4 | .4918 | .4920 | .4922 | .4925 | .4927 | .4929 | .4931 | .4932 | .4934 | .4936 |
| 2.5 | .4938 | .4940 | .4941 | .4943 | .4945 | .4946 | .4948 | .4949 | .4951 | .4952 |
| 2.6 | .4953 | .4955 | .4956 | .4957 | .4959 | .4960 | .4961 | .4962 | .4963 | .4964 |
| 2.7 | .4965 | .4966 | .4967 | .4968 | .4969 | .4970 | .4971 | .4972 | .4973 | .4974 |
| 2.8 | .4974 | .4975 | .4976 | .4977 | .4977 | .4978 | .4979 | .4979 | .4980 | .4981 |
| 2.9 | .4981 | .4982 | .4982 | .4983 | .4984 | .4984 | .4985 | .4985 | .4986 | .4986 |
| 3.0 | .4987 | .4987 | .4987 | .4988 | .4988 | .4989 | .4989 | .4989 | .4990 | .4990 |

Source: Abridged from Table 1 of A. Hald, *Statistical Tables and Formulas* (New York: Wiley), 1952. Reproduced by permission of A. Hald and the publisher, John Wiley & Sons, Inc.

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-w^2/2}\, dw \quad \longleftrightarrow$$



# Conjugate Priors

| **Prior $h(\Theta)$** | **Likelihood $g(x\mid\Theta)$** | **Posterior $f(\Theta\mid x)$** |
|---|---|---|
| $B(\alpha,\beta)$: $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$ | *Binomial*: $\binom{n}{x}p^x(1-p)^{n-x}$ | $B(\alpha+x,\beta+n-x)$ |
| $\Gamma(\alpha,\beta)$: $\frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^{\alpha}}$ *if* $\theta\geq 0$ | *Poisson*: $\frac{e^{-\theta}\theta^x}{x!}$ | $\Gamma\left(\alpha+x,\frac{\beta}{n\beta+1}\right)$ |
| $N(\mu,\tau^2)$: $\frac{Exp\left(-\frac{(\theta-\mu)^2}{2\tau^2}\right)}{\tau\sqrt{2\pi}}$ | $N(\theta,\sigma^2)$: $\frac{Exp\left(-\frac{(x-\theta)^2}{2\sigma^2}\right)}{\sigma\sqrt{2\pi}}$ | $N\left(\frac{\sigma^2\mu+\tau^2}{\sigma^2+\tau^2},\frac{\sigma^2\tau^2}{\sigma^2+\tau^2}\right)$ |