<u>This week</u>: Dr. Osonde Osoba

- Expectations and Moments
  - Theory/definition, properties
  - means and Variances (μ, σ^2)
- Examples of (μ, σ^2)

<u>H/W 5:</u>

L-G: 4.39, 4.40, 4.53 – 4.57, 4.70

I/ <u>Expectations</u>:

Recall the definition of a random variable

$$X : (\Omega, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

as a measurable map (PAM) between $(\Omega, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

The expectation of X is the integral

$$\boxed{E[X] = \int_\Omega X(\omega)\, dP(\omega)}$$

with respect to the the probability measure, P, on the probability space $(\Omega, \mathcal{A}, P)$.

This integral on Ω is equivalent to the following integral on $\mathbb{R}$:

$$E[X] = \int_{\mathbb{R}} x \, dF_X(x)$$

$$E[X] = \int_{\mathbb{R}} x \, f_X(x) \, dx \quad \text{[for absolutely continuous distribution]}$$

$$E[X] = \sum_k k \, P_X(k) \quad \text{[for discrete distribution]}$$

$$(E[X] \equiv \mu_X)$$

vs mode vs median.

Properties of $E[\cdot]$:

$E[\cdot]$ is an operator (a function on functions).

If a, b are scalars and X, Y are r.v.s

(a) $E[aX + bY] = a\,E[X] + b\,E[Y]$

(Linearity, homogeneity)

(b) $[X \geq Y] \Rightarrow [E[X] \geq E[Y]]$

Define:

(i) k^{th} order Moment

$$E[X^k] = \int_{-\infty}^{\infty} x^k f_X(x)\, dx$$

(ii) Variance σ_X^2 / $Var[X]$

$$\sigma_X^2 = E\left[(X - E[X])^2\right] = E[X^2] - (E[X])^2$$

Qu: $E[X^2] \geq E^2[X]$?

(iii) Covariance σ_{XY} / $Cov[X, Y]$

$$\sigma_{XY} = E\left[(X-\mu_X)(Y-\mu_Y)\right] = E[XY] - \mu_X \cdot \mu_Y$$

$$\begin{aligned} Var[aX+bY] &= E[(aX+bY)^2] - \left(E[(aX+bY)]\right)^2 \\ &= E[a^2X^2 + b^2Y^2 + 2ab\,XY] - (a\mu_X + b\mu_Y)^2 \\ &= a^2 E[X^2] + b^2 E[Y^2] + 2ab\, E[XY] - a^2\mu_X^2 - b^2\mu_Y^2 - 2ab\,\mu_X\mu_Y \end{aligned}$$

$$\boxed{Var[aX+bY] = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,\sigma_{XY}}$$

i.e $E[\cdot]$ is linear while $Var[\cdot]$ is quadratic

$$E[XY] = \iint x\, y\, f_{XY}(x,y)\, dx\, dy$$

$X\ Y$ indep. $\Rightarrow f_{XY}(x,y) = f_X(x)\, f_Y(y)$

$$\Rightarrow E[XY] = \iint x\, y\, f_X(x) \cdot f_Y(y)\, dx\, dy$$

$$= \left[\int x\, f_X(x)\, dx\right]\left[\int y\, f_Y(y)\, dy\right]$$

$$= \mu_X \cdot \mu_Y$$

$\therefore\ \sigma_{XY} = 0$ if X and Y are independent.

$$\therefore\ \mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = n \cdot \mathrm{Var}[X]$$

if $\{X_i\}_{i=1}^{n}$ are independent + identically distributed

While $E\left[\sum_i^n X_i\right] = \sum_i^n E[X_i]$ always.

Claim: if $k \le m$ $(k, m \in \mathbb{Z}^+)$

then $E[|X^m|] < \infty \Rightarrow E[|X^k|] < \infty$

Proof:

$$\int_{-\infty}^{\infty} |x|^k f_X(x)\, dx = \int_{|x|\le 1} |x|^k f_X(x)\, dx + \int_{|x|>1} |x|^k f_X(x)\, dx$$

$\left[\begin{array}{l}|x|^m \ge |x|^k \\ \text{for } |x|>1\end{array}\right]$

$$\le \int_{|x|\le 1} |x|^k f_X(x)\, dx + \int_{|x|>1} |x|^m f_X(x)\, dx$$

$\left[\begin{array}{l}|x|^k \le 1 \\ \forall\, |x| \le 1\end{array}\right]$

$$\le \int_{|x|\le 1} f_X(x)\, dx + \int_{|x|>1} |x|^m f_X(x)\, dx$$

$$\le \int_{-\infty}^{\infty} f_X(x)\, dx + \int_{|x|>1} |x|^m f_X(x)\, dx$$

$$\Rightarrow E[|X|^k] \le 1 + \int_{-\infty}^{\infty} |x|^m f_X(x)\, dx$$

$$= 1 + E[|X|^m]$$

$$\therefore E[|X|^k] < \infty \quad \text{since} \quad E[|X|^m] < \infty$$

e.g. $\sigma_X^2 < \infty \Rightarrow |\mu_X| < \infty$

Examples of $E[\cdot]$ and $Var[\cdot]$:

i/ $(\Omega, \mathcal{Q}, P)$; $A \in \mathcal{Q}$

$$X(\omega) = I_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \in A^c \end{cases}$$

$$E[X] = \int X(\omega)\, dP(\omega) = 1 \cdot P(A) + 0 \cdot P(A^c)$$

$$E[I_A] = P(A)$$

$$Var[I_A] = E[I_A^2] - (E[I_A])^2$$

$$= \left(1^2 \cdot P(A) + 0^2 \cdot P(A^c)\right) - (P(A))^2$$

$$= P(A) - (P(A))^2$$

$$Var[I_A] = P(A)\,[1 - P(A)]$$

$$(\mu_X, \sigma_X^2) = (P(A),\; P(A)[1 - P(A)])$$

[cf. Bernoulli rv]

ii) $X \sim \exp(\theta)$:

$$E[X^k] = \int_0^\infty x^k \cdot 1/\theta \cdot \exp(-x/\theta)\, dx$$
$$= \theta^{k-1} \int_0^\infty (x/\theta)^k \exp(-x/\theta)\, dx$$

$[y = x/\theta]$

$$\Rightarrow \quad E[X^k] = \theta^{k-1} \cdot \theta \cdot \int_0^\infty y^k \cdot \exp(-y)\, dy$$
$$= \theta^k \cdot \Gamma(k+1) \qquad \left[\because \Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z} dz\right]$$

$$k \in \mathbb{Z}^+ \Rightarrow \boxed{E[X^k] = \theta^k \cdot k!}$$

$[\because \Gamma(n) = (n-1)!]$

e.g. $Y \sim \exp(1) \Rightarrow E[Y^k] = k!$

$$\Rightarrow \quad \mu_X = \theta$$
$$\sigma_X^2 = E[X^2] - \mu_X^2 = (2!)\theta^2 - \theta^2$$
$$\sigma_X^2 = \theta^2$$

iii) $X \sim \Gamma(\alpha, \theta)$:

$$E[X^k] = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^k/\theta^\alpha \cdot x^{\alpha-1} \exp(-x/\theta)\, dx$$
$$= \frac{1}{\Gamma(\alpha)\theta^\alpha} \cdot \int_0^\infty x^{k+\alpha-1} \exp(-x/\theta)\, dx$$

$[y = +x/\theta]$

$$\Rightarrow E[X^k] = (\Gamma(\alpha)\,\theta^\alpha)^{-1} \cdot \theta \cdot \theta^{k+\alpha-1} \int_0^\infty y^{k+\alpha-1} \cdot \exp(-y)\,dy$$

$$= \frac{\theta^{k+\alpha}}{\Gamma(\alpha)\,\theta^\alpha} \cdot \Gamma(\alpha+k)$$

$$\boxed{E[X^k] = \theta^k \cdot \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}}$$

$$\mu_X = E[X] = \theta^1 \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\theta \cdot \alpha \cdot \Gamma(\alpha)}{\Gamma(\alpha)}$$

$$\boxed{\mu_X = \alpha \cdot \theta}$$

$$\sigma_X^2 = \mathrm{Var}[X] = E[X^2] - \mu_X^2$$

$$E[X^2] = \theta^2 \cdot \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{\theta^2 \cdot (\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{(\alpha+1)\theta^2 \cdot \Gamma(\alpha) \cdot \alpha}{\Gamma(\alpha)}$$

$$= (\alpha)\,\theta^2 \cdot (\alpha+1)$$

$$\Rightarrow \sigma_X^2 = (\alpha)\,\theta^2(\alpha+1) - \alpha^2\theta^2$$

$$= \cancel{\alpha^2\theta^2} + \alpha\theta^2 - \cancel{\alpha^2\theta^2}$$

$$\boxed{\sigma_X^2 = \alpha\,\theta^2}$$

iv $X \sim \text{Poisson}(\lambda)$

$$E[X] = e^{-\lambda} \cdot \sum_{k=1}^{\infty} k \cdot \lambda^k / k!$$

since 0 @ $k=0$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \lambda^k / (k-1)! = e^{-\lambda} \cdot \lambda \cdot \sum_{k=1}^{\infty} \lambda^{k-1} / (k-1)!$$

$$= e^{-\lambda} \cdot \lambda \sum_{j=0}^{\infty} \lambda^j / j! = e^{-\lambda} \cdot \lambda \cdot e^{\lambda}$$

$$\mu_X = \lambda$$

$$E[X(X-1)] = e^{-\lambda} \sum_{k=0}^{\infty} k(k-1) \lambda^k / k!$$

$$= e^{-\lambda} \sum_{k=2}^{\infty} \lambda^k \cdot k(k-1) / k!$$

$$= e^{-\lambda} \sum_{k=2}^{\infty} \lambda^k / (k-2)!$$

$$= \lambda^2 \cdot e^{-\lambda} \cdot \sum_{k=2}^{\infty} \lambda^{k-2} / (k-2)!$$

$$= \lambda^2 \cdot e^{-\lambda} \cdot \sum_{j=0}^{\infty} \lambda^j / j! = \lambda^2 \cdot e^{-\lambda} \cdot e^{\lambda}$$

$$E[X(X-1)] = \lambda^2$$

$$\Rightarrow \quad E[X^2] = \lambda^2 + E[X] = \lambda^2 + \lambda$$

$$\sigma_X^2 = E[X^2] - \mu_X^2 = \lambda^2 + \lambda - \lambda^2$$

$$\sigma_X^2 = \lambda$$

Stein's Characterization for Poisson:

$$E[X \cdot f(X)] = \lambda E[f(X+1)]$$

Aside

Stein's Characterization for $X \sim \text{Poisson}(\lambda)$:

Claim: $\forall f(\cdot)$ for which the expectation exists.

$$[X \sim \text{Poisson}(\lambda)] \implies \left[E[X \cdot f(X)] = \lambda \cdot E[f(X+1)]\right]$$

Proof:

$$X \sim \text{Poisson}(\lambda) \Rightarrow E[X \cdot f(X)] = \sum_{k=0}^{\infty} k \cdot f(k) \cdot e^{-\lambda} \cdot \lambda^k / k!$$

$[\because k \cdot f(k)|_{k=0} = 0]$ $$= e^{-\lambda} \sum_{k=1}^{\infty} k \cdot f(k) \cdot \lambda^{k-1} \cdot \lambda / k!$$

$$= \lambda \cdot e^{-\lambda} \sum_{k=1}^{\infty} f(k)\, \lambda^{k-1}/(k-1)!$$

$[j = k-1]$ $$= \lambda \cdot e^{-\lambda} \sum_{j=0}^{\infty} f(j+1) \cdot \lambda^j / j!$$

$[\text{def}^n : E[\cdot]]$ $$\therefore E[X \cdot f(X)] = \lambda \cdot E[f(X+1)] \;/\!/$$

e.g.: i) $f(x) = 1 \quad \forall x$

$$\Rightarrow E[X \cdot f(X)] = \lambda E[f(X+1)] = \lambda \cdot E[1] = \lambda \cdot 1$$

$$\text{i.e. } E[X] = \lambda \;/\!/$$

ii) $f(x) = (x-1) \quad \forall x$

$$\Rightarrow E[X \cdot f(X)] = \lambda \cdot E[f(X+1)] = \lambda \cdot E[X] = \lambda \cdot \lambda$$

$$\text{i.e. } E[X \cdot (X-1)] = \lambda^2 \;/\!/$$

iii) $f(x) = (x-1)(x-2)$

$$\Rightarrow E[X \cdot f(X)] = \lambda E[f(X+1)] = \lambda \cdot E[X \cdot (X-1)] = \lambda \cdot \lambda^2 = \lambda^3 \;/\!/$$

By induction: $$\boxed{E[X \cdot (X-1) \cdots (X-k+1)] = \lambda^k}$$ $\forall k \in \mathbb{Z}^+$ for $X \sim \text{Poisson}(\lambda)$

v. $X \sim \text{Geometric}(p)$

$p \in (0,1]$ $\quad q = (1-p) \in [0,1) \leftarrow \subset$ ROC for Geometric series

$$E[X] = \sum_{k=1}^{\infty} p \cdot k \cdot (1-p)^{k-1}$$

$$= p \sum_{k=1}^{\infty} k \cdot q^{k-1}$$

$$E[X] = p \sum_{k=1}^{\infty} \partial q^k / \partial q$$

$$= p \cdot \partial/\partial q \left(\sum_{k=1}^{\infty} q^k \right) \qquad [\because q \in \text{ROC of Geometric series}]$$

$$= p \cdot \partial/\partial q \left(\frac{q}{1-q} \right)$$

$$= p \left(\frac{(1-q)+q}{(1-q)^2} \right) = p \cdot 1/p^2$$

$$\therefore \boxed{E[X] = 1/p}$$

$$E[X(X-1)] = \sum_{k=1}^{\infty} k \cdot (k-1) \cdot p \cdot q^{k-1}$$

$$= \sum_{k=1}^{\infty} p \cdot q \cdot k(k-1)\, q^{k-2}$$

$$= p \cdot q \sum_{k=1}^{\infty} q^{k-2} \cdot k \cdot (k-1) = p \cdot q \sum_{k=1}^{\infty} \frac{\partial^2 (q^k)}{\partial q^2}$$

$$= p \cdot q \; \partial^2/\partial q^2 \left(\sum_{k=1}^{\infty} q^k \right)$$

$$= p \cdot q \; \partial^2/\partial q^2 \left(\frac{q}{1-q} \right) = 2 p \cdot q \cdot \frac{1}{(1-q)^3}$$

$$\therefore E[X(X-1)] = 2q/p^2$$

$$\Rightarrow E[X^2] - E[X] = 2q/p^2$$

$$E[X^2] = 2q/p^2 + 1/p$$

$$E[X^2] = \frac{2q+p}{p^2}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{2q+p}{p^2} - \frac{1}{p^2}$$

$$\therefore \boxed{\sigma_X^2 = q/p^2}$$

Ex [The Coupon Collector's Problem]:

Qu: A magazine features one of m coupons in each release. What is the average # of magazines a collector must buy to get all m coupons?

Ans: T = # of magazines bought until last coupon.

$$T = X_1 + X_2 + \cdots + X_m$$

X_i = # of magazines bought before getting i^{th} coupon

$\therefore\ X_i \sim \text{Geometric}(p_i)$

$p_i = 1 - \left(\frac{i-1}{m}\right)$ [Why?]

[Since the probability of getting a new type of coupon reduces as we have more coupons. $\left(p_1 = 1,\ p_2 = \frac{m-1}{m}, \ldots, p_m = 1/m\right)$]

$$E[T] = \sum_{i=1}^{m} E[X_i] = \sum_{i=1}^{m} \left[1 - \left(\frac{i-1}{m}\right)\right]^{-1}$$

$$= \sum_{i=1}^{m} \frac{m}{m-i+1} = m \sum_{i=1}^{m} \frac{1}{m-i+1}$$

Let $n = m-i+1$. $i = 1 \Rightarrow n = m$, $i = m \Rightarrow n = 1$

$$\therefore\ \boxed{E[T] = m \cdot \sum_{n=1}^{m} 1/n}$$

vi $X \sim Beta(\alpha, \beta)$

Recall

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$E[X^k] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}\int_0^1 x^k\cdot x^{\alpha-1}\cdot(1-x)^{\beta-1}\,dx$$

$$= \frac{1}{B(\alpha,\beta)}\int_0^1 x^{(\alpha+k)-1}(1-x)^{\beta-1}\,dx$$

$$\boxed{E[X^k] = \frac{B(\alpha+k,\beta)}{B(\alpha,\beta)}}$$

$$\Rightarrow E[X] = \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+1)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\cdot\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}$$

$$= \frac{\alpha\cdot\Gamma(\alpha)\cdot\Gamma(\alpha+\beta)}{(\alpha+\beta)\cdot\Gamma(\alpha+\beta)\cdot\Gamma(\alpha)}$$

$$\boxed{E[X] = \frac{\alpha}{\alpha+\beta}}$$

$$E[X^2] = \frac{B(\alpha+2,\beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+2)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}$$

$$= \frac{(\alpha+1)\cdot\Gamma(\alpha+1)\cdot\Gamma(\alpha+\beta)}{(\alpha+\beta+1)\,\Gamma(\alpha+\beta+1)\cdot\Gamma(\alpha)}$$

$$= \frac{(\alpha+1)\cdot(\alpha)\cdot\cancel{\Gamma(\alpha)}\cdot\cancel{\Gamma(\alpha+\beta)}}{(\alpha+\beta+1)\cdot(\alpha+\beta)\cdot\cancel{\Gamma(\alpha+\beta)}\cdot\cancel{\Gamma(\alpha)}}$$

$$E[X^2] = \frac{\alpha\cdot(\alpha+1)}{(\alpha+\beta)\cdot(\alpha+\beta+1)}$$

$$\Rightarrow \sigma_x^2 = \frac{\alpha\cdot(\alpha+1)}{(\alpha+\beta)\cdot(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha\cdot(\alpha+1)\cdot(\alpha+\beta) - \alpha^2\cdot(\alpha+\beta+1)}{(\alpha+\beta)^2\cdot(\alpha+\beta+1)}$$

$$= \frac{\alpha^3+\alpha\beta+\alpha^2\beta+\alpha^2-\alpha^3-\alpha^2\beta-\alpha^2}{(\alpha+\beta)^2\cdot(\alpha+\beta+1)}$$

$$\boxed{\sigma_x^2 = \frac{\alpha\beta}{(\alpha+\beta)^2\cdot(\alpha+\beta+1)}}$$

vii $X \sim \text{Normal}(\mu, \sigma_X^2)$

Suppose $Z \sim N(0,1)$

i.e. $E[Z] = 0$

$\sigma_Z^2 = 1$

$X = (\sigma_X \cdot Z + \mu) \sim N(\mu, \sigma_X^2)$

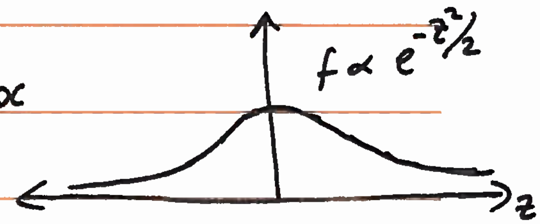
$E[X] = \sigma_X E[Z] + \mu = \mu$

$Var[X] = \sigma_X^2 \, Var[Z] = \sigma_X^2$

Recall:

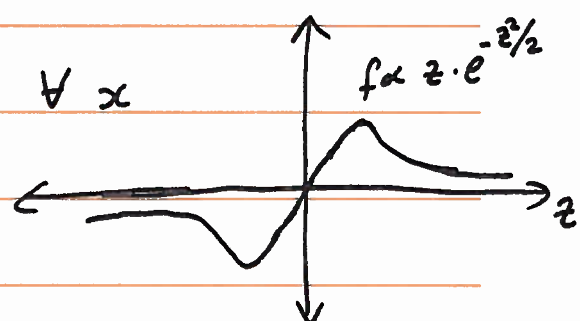
$f(x)$ is an even function

$\iff \quad f(-x) = f(x) \quad \forall x$



$f(x)$ is an odd function

$\iff \quad f(-x) = -f(x) \quad \forall x$



f odd $\Rightarrow \int_{\mathbb{R}} f(x)\,dx = 0$

f even $\Rightarrow \int_{\mathbb{R}} f(x)\,dx = 2\int_0^{\infty} f(x)\,dx$.

The standard Normal pdf, $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, is an even function. $z^k \cdot f_Z(z)$ is also even when k is even. $z^k \cdot f_Z(z)$ is odd when k is odd.

$$E[Z^k] = \int_{\mathbb{R}} z^k f_Z(z)\, dz \qquad k \in \mathbb{Z}^+$$

$$E[Z^k] = \begin{cases} 0 & k \text{ odd} \\ \prod_{j=1}^{k/2} (k-(2j+1)) & k \text{ even} \end{cases}$$

And in general:

$X \sim N(\mu, \sigma^2)$

$$\Rightarrow \quad E[(X-\mu)^k] = \begin{cases} 0 & k \text{ odd} \\ \sigma^k \cdot \prod_{j=1}^{k/2} [k-(2j+1)] & k \text{ even} \end{cases}$$

Proof: (a) k odd

$$\Rightarrow \quad E[Z^k] = \int_{\mathbb{R}} z^k \cdot f_Z(z)\, dz$$

$z^k \cdot f_Z(z) \propto z^k \cdot \exp(-z^2/2)$, an odd function

$$(-z)^k \cdot f_Z(-z) \propto (-1)^k \cdot z^k \cdot \exp(-(-z)^2/2)$$

$$= (-1)\, z^k \cdot \exp(-z^2/2) = -z^k f_Z(z)$$

$$\therefore \quad \int_{\mathbb{R}} z^k \cdot f_Z(z)\, dz = 0$$

$$E[Z^k] = 0$$

(b) $\underline{k \text{ even}}$

$$E[z^k] = \int_{-\infty}^{\infty} z^k \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp(-z^2/2)\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{k-1} \cdot z \cdot \exp(-z^2/2)\, dz$$

Integration by parts: $\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$

$u = z^{k-1}$ | $v = -e^{-z^2/2}$

$du = (k-1)\, z^{k-2}\, dz$ | $dv = z \cdot \exp(-z^2/2)\, dz$

$$\Rightarrow E[z^k] = \frac{1}{\sqrt{2\pi}}\left[-z^{k-1} \cdot e^{-z^2/2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -e^{-z^2/2} \cdot (k-1) \cdot z^{k-2}\, dz \right]$$

$$\lim_{z \to \pm\infty} z^{k-1} \cdot e^{-z^2/2} = 0$$

$$\Rightarrow E[z^k] = (k-1) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z^{k-2} \cdot e^{-z^2/2}\, dz$$

$$\therefore \boxed{E[z^k] = (k-1)\, E[z^{k-2}]}$$

k even $\Leftrightarrow$ $(k-2)$ even. So same argument for $E[z^{k-2}] = (k-3) \cdot E[z^{k-4}]$ all the way down to $E[z^0] = 1$

$$\Rightarrow E[z^k] = (k-1)(k-3)(k-5) \cdots 5 \cdot 3 \cdot 1$$

$$\therefore \boxed{E[z^k] = \prod_{j=1}^{k/2} [k - (2j+1)]} \quad \text{for } k \text{ even}$$