This Week : Dr. Osonde Osoba

- Multiple Random Variables
  - Distribution functions
  - Marginal and Conditionals
  - RV Independence and Implications
  - Expectations of multiple rvs
  - Bayes Theorem and Total Probability for rvs [SIT]
- Transformations of rvs
  - $F_X(\cdot)$ approach
  - General Jacobian formula approach

H/W 7 :

5.8, 5.12, 5.13, 5.25, 5.26, 5.35

4.90 — 4.94

I/ Multiple Random Variables:

So far we have described random variables in isolation. e.g knowing $f_X(x)$ for $X(\omega)$ completely describes X.

Working with ensembles of random variables requires an extra level of abstraction to handle interdependencies. e.g. knowing f_X for X and f_Y for Y is not enough to make meaningful statements about events like: $\{(X,Y) \in A \times B\}$ because X and Y may have a nontrivial dependence.

We need a joint distribution specification. The joint CDF:

$$\boxed{\begin{aligned} F_{XY}(x,y) &= P(X \in (-\infty, x],\ Y \in (-\infty, y]) \\ &= P(X \le x,\ Y \le y) \end{aligned}}$$

is the most fundamental joint description for (X,Y)

Assuming absolute continuity, we get a joint PDF:

$$\boxed{f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}}$$

We also get expectations: (for measurable fxns $g(x,y)$)

$$\boxed{E_{XY}[g(X,Y)] = \iint g(x,y) \cdot f_{XY}(x,y)\, dx\, dy}$$

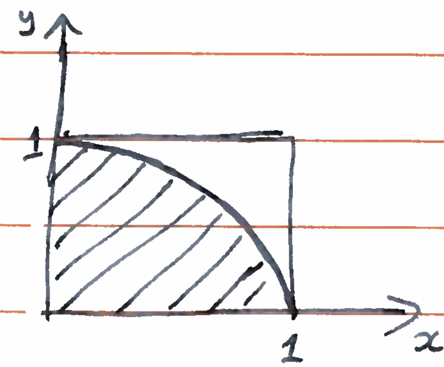
We can now write for $D \in \mathcal{B}(\mathbb{R} \times \mathbb{R})$

$$\boxed{P((X,Y) \in D) = \iint_D f_{XY}(x,y)\, dx\, dy}$$

e.g.: $f_{XY}(x,y) = \begin{cases} 0 & (x,y) \notin [0,1] \times [0,1] \\ 1 & 0 \le x, y \le 1 \end{cases}$

$D = \{(x+y) \mid x^2+y^2=1\}$

$P((X,Y) \in D) = ??$



Ans: $P(D) = \iint_D f_{XY}(x,y)\, dx\, dy$

$= \iint_D 1 \, dx\, dy$

$dx\, dy = r\, dr\, d\theta \;;\; \theta \in (0, \pi/2), \; r \in (0,1]$

$\Rightarrow P(D) = \int_0^{\pi/2} \int_0^1 r\, dr\, d\theta$

$= \pi/2 \int_0^1 r\, dr$

$= \pi/4 \; r \Big|_0^1$

$P(D) = \pi/4$

[Sometimes used as a probabilistic method for estimating π]

(cf. Monte Carlo methods)

The joint descriptions, f_{XY}, F_{XY}, subsume the marginal descriptions:

$$f_X(x) = \int_{\mathbb{R}} f_{XY}(x,y)\,dy \qquad F_X(x) = \lim_{y\to\infty} F_{XY}(x,y)$$

$$f_Y(y) = \int_{\mathbb{R}} f_{XY}(x,y)\,dx \qquad F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y)$$

They also allow for conditional descriptions

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \qquad f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

or $f_{XY}(x,y) = f_{X|Y}(x|y)\cdot f_Y(y) = f_{Y|X}(y|x)\cdot f_X(x)$

By arguments with infinitesimals, we get Bayes Theorem

$$\boxed{f(x|y) = \frac{f(x)\cdot f(y|x)}{\int f(x)\, f(y|x)\,dx}}$$

based on the Total Probability identity

$$\boxed{\begin{aligned} f(y) &= \int f_{XY}(x,y)\,dx \\ &= \int f(x)\cdot f(y|x)\,dx \end{aligned}}$$

[TP]

The last decomposition for $f_Y(y)$ also gives

$$\boxed{P(X \in A) = \int P(X \in A \mid Y = y) \cdot f_Y(y)\, dy}$$

$$\boxed{P((X,Y) \in D) = \int P((X,Y) \in D \mid Y = y) \cdot f_Y(y)\, dy}$$

where $P((X,Y) \in D \mid Y = y) = \int_D f_{X|Y}(x|y)\, dx$

RV Independence :

X, Y independent

$$\boxed{\begin{aligned}
&\iff P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B) \quad \forall A, B \in \mathbb{B}(\mathbb{R}) \\
&\iff P(X \in (-\infty, x], Y \in (-\infty, y]) = P(X \in (-\infty, x]) \cdot P(Y \in (-\infty, y]) \\
&\iff F_{XY}(x, y) = F_X(x) \cdot F_Y(y) \\
&\iff f_{XY}(x, y) = f_X(x) \cdot f_Y(y) \\
&\iff f_{X|Y}(x|y) = f_X(x) \;;\; f_{Y|X}(y|x) = f_Y(y)
\end{aligned}}$$

Also X, Y independent ($\forall$ measurable fxns g, h)

$\Rightarrow E_{XY}[g(X) \cdot h(Y)] = E_X[g(X)] \cdot E_Y[h(Y)]$

This is a very stringent condition. e.g. suppose $E[X] = 0$. then $\forall$ meas. $g(\cdot)$

$$E[X \cdot g(Y)] = E[X] \cdot E[g(Y)] = 0 \quad \forall g(\cdot) \;!!!$$

Also if X, Y independent

$$\Rightarrow P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\, dx \overset{\text{Ind.}}{=} \int_A f_X(x)\, dx$$

$$P(X \in A \mid Y=y) = P(X \in A)$$

"SIT" Technique for calculating probabilities:

(1) T_p: Use Total Probability

(2) S: Substitute* x for r.v. X if $X = x$

(3) I: Use Independence of X, Y if applicable

e.g [T_p]

$Y \sim \exp(1)$; $X \sim U(0,1)$; $X \perp\!\!\!\perp Y$

$P(Y > X) = ???$

Ans

$$P(Y>X) \overset{T_p}{=} \int P(Y>X \mid X=x) \cdot f_X(x)\, dx$$

$$\overset{S}{=} \int P(Y>x \mid X=x) \cdot f_X(x)\, dx$$

$$\overset{\text{Ind}}{=} \int P(Y>x) \cdot f_X(x)\, dx$$

$$= \int_0^1 \exp(-x) \cdot 1\, dx$$

$$\therefore P(Y>X) = 1 - \exp(-1)$$

*Refer to Gubner Example 13.23 for a more complete exposition/proof of the substitution Law e.g.: [Tp] X, Y, Z : iid $U(0,1)$

$P(X \geq YZ) = ?$

Ans:

$$P(X \geq YZ) \overset{Tp}{=} \iint P(X \geq yz \mid Y=y, Z=z) \cdot f_{YZ}(y,z)\, dy\, dz$$

$$\overset{ind.\,YZ}{=} \int_0^1 \int_0^1 P(X \geq yz \mid Y=y, Z=z) \cdot f_{Y}(y) \cdot f_Z(z)\, dy\, dz$$

$$\overset{ind\,X}{=} \int_0^1 \int_0^1 P(X \geq yz) \cdot f_Y(y) \cdot f_Z(z)\, dy\, dz$$

$$= \iint [1 - F_X(yz)] \cdot f_Y(y)\, f_Z(z)\, dy\, dz$$

$$= \iint (1 - y \cdot z) \cdot f_Y(y) \cdot f_Z(z)\, dy\, dz$$

$$= \left[\int_0^1 y\, f_Y(y)\right]$$

$$= \int_0^1 \int_0^1 (1 - yz)\, dy\, dz$$

$$= \int_0^1 \left(y - y^2 z/2\right)\Big|_0^1 dz$$

$$= \int_0^1 \left(1 - z/2\right) dz$$

$$= z - z^2/4 \Big|_0^1$$

$$= 1 - 1/4$$

$$P(X \geq YZ) = 3/4$$

e.g : [Bayes Thm for rvs]

Suppose $X \sim b(n,p)$ $p \sim Beta(\alpha, \beta)$

Qu find $f(p|X=k)$

Ans: Recall $(b(n,p), Beta(\alpha,\beta))$ are conjugate $\Rightarrow p|X=k \sim Beta(\cdot,\cdot)$

Ans :

$$f(p|x) = \frac{f(X=k|p)\cdot h(p)}{\int f(X=k|p)\cdot h(p)\, dp}$$

$$= \frac{\cancel{\binom{n}{k}} \cdot \cancel{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}} \cdot p^k (1-p)^{n-k} \cdot p^{\alpha-1} \cdot (1-p)^{\beta-1}}{\cancel{\binom{n}{k}} \cdot \cancel{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)}} \cdot \int_0^1 p^{\alpha+k-1} \cdot (1-p)^{\beta+n-k-1} dp}$$

$$f(p|X) = \frac{p^{\alpha+k-1} \cdot (1-p)^{\beta+n-k-1}}{\int_0^1 \dots \, dp}$$

$$\therefore \quad f(p|X=k) = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+k)\,\Gamma(\beta+n-k)} \, p^{\alpha+k-1} \cdot (1-p)^{\beta+n-k-1}$$

$$p|X=k \sim Beta(\alpha+k, \beta+n-k)$$

Qu: Given our updated pdf $f(p|X=k)$ of p given observed data $X=k$, What is our current "best" estimate, $\hat{\theta}$, for the value of p?

Ans: This depends on how you define "best"/"optimal" estimates. Bayes-optimal estimates define $\hat{\theta}$ as the estimate that minimizes an average ~~loss~~ loss : $\boxed{\hat{\theta} = \underset{\theta}{\operatorname{argmin}}\ E_{p|x}[L(\theta;p)]}$

eg: Pick the optimal estimate $\hat{\theta}$ for p that minimizes the average square estimation error $l(\theta;p) = (p-\theta)^2$ between the current estimate, θ, and true value p.

ⓐ The loss function / error function is

$l(\theta;p) = (p-\theta)^2$ [Squared Error Loss]

Average loss $R(\theta) = E_{p|x}\left[l(\theta;p)\right]$ [Risk function]

$\Rightarrow R(\theta) = E\left[(p-\theta)^2\right] = \int_0^1 (p-\theta)^2 \cdot f(p|x)\,dp$

$\hat{\theta} = \underset{\theta}{\arg\min}\, R(\theta) = \underset{\theta}{\arg\min} \left\{ E\left[(p-\theta)^2\right] \right\} = \underset{\theta}{\arg\min}\left\{E[p^2] - 2\theta E[p] + \theta^2\right\}$

$\Rightarrow \partial R / \partial \theta = -2E[p] + 2\theta = 0$

$\Rightarrow \boxed{\hat{\theta} = E_{p|x}[p]}$

ⓑ $l(\theta;p) = \begin{cases} 0 & \theta = p \\ 1 & \theta \neq p \end{cases} = 1 - \delta(p-\theta)$ [0-1 loss]

$\Rightarrow R(\theta) = E_{p|x}\left[1 - \delta(p-\theta)\right] = 1 - \int \delta(p-\theta) \cdot f_{p|x}(p|x)\,dp$

$R(\theta) = 1 - f(p|x)\big|_{p=\theta}$

$\hat{\theta} = \underset{\theta}{\arg\min}\, R(\theta) \Leftrightarrow \hat{\theta} = \underset{\theta}{\arg\max}\, f(p|x)$

i.e. $\boxed{\hat{\theta} = \text{mode}\left(f(p|x)\right)}$

ⓒ $l(\theta;p) = |p-\theta|$ [absolute error loss]

$\Rightarrow R(\theta) = \int_0^1 |p-\theta|\, f(p|x)\,dp$

$\hat{\theta} = \arg\min\{R(\theta)\} = \arg\min\left\{ \int_0^\theta (\theta-p)\, f_{p|x}\,dp + \int_\theta^1 (p-\theta)\, f_{p|x}\,dp \right\}$

i.e. $\hat{\theta}$ such that $\int_0^{\hat{\theta}} f_{p|x}\,dp = \int_{\hat{\theta}}^1 f_{p|x}\,dp$ $\therefore$ $\boxed{\hat{\theta} = \text{median}(f(p|x))}$

eg: $[E_{xy}[\cdot]]$

X is independent of Y

$E[XY] = ?$ $\quad Var[XY] = ?$

ans:

$$E_{XY}[XY] = \iint xy \, f_{XY}(x,y)\, dx\, dy$$

$$\stackrel{ind.}{=} \iint x\cdot y \; f_X(x)\cdot f_Y(y)\, dx\, dy$$

$$= \left[\int x\cdot f_X(x)\, dx\right]\left[\int y\, f_Y(y)\, dy\right]$$

$$= E_X[X]\; E_Y[Y]$$

$$Var[XY] = E[(XY)^2] - (E[XY])^2$$

$$\stackrel{ind.}{=} E[X^2]\cdot E[Y^2] - E^2[X]\cdot E^2[Y]$$

$$= (\sigma_X^2+\mu_X^2)\cdot(\sigma_Y^2+\mu_Y^2) - \mu_X^2\cdot\mu_Y^2$$

$$Var[XY] = \sigma_X^2\cdot\sigma_Y^2 + \mu_X^2\,\sigma_Y^2 + \mu_Y^2\,\sigma_X^2$$

Suppose $\mu_X = \mu_Y = 0$

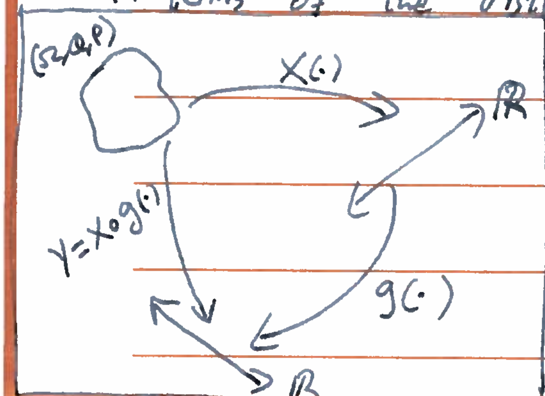
$\Rightarrow$ $Var[XY] = \sigma_X^2\cdot\sigma_Y^2$

II/ Transformation of Random Variables

Suppose $X : \Omega \to \mathbb{R}$ is a random variable and $g : \mathbb{R} \to \mathbb{R}$ is a measurable function.

Define a derived rv : $Y = g(X)$.

How do we express the distribution law for Y in terms of the distribution of X which we know?



We just use the preimage for g to convert Borel sets for Y into Borel set for X which we can measure:

$$\boxed{P(Y \in B) = P(g \circ X \in B) = P(X \in g^{-1}(B))} \; *$$

e.g. $Z \sim C(0,1)$; $W = g(Z) = |Z|$

$f_W(\omega) = ?$

Ans :

$$f_Z(z) = \frac{1}{\pi(1+z^2)}$$

$$F_W(\omega) = P(W \le \omega)$$
$$= P(g(Z) \le \omega)$$
$$= P(|Z| \le \omega)$$
$$= P(-\omega \le Z \le \omega) \quad *$$
$$= F_Z(\omega) - F_Z(-\omega)$$

$$\Rightarrow \partial_\omega F_W(\omega) = \partial_\omega F_Z(\omega) - \left(-\partial_\omega F_Z(-\omega)\right)$$

$$\Rightarrow f_W(\omega) = f_Z(\omega) + f_Z(-\omega)$$
$$= f_Z(\omega) + f_Z(\omega) \quad \leftarrow [f_Z \text{ even}]$$

$$f_W(\omega) = \begin{cases} \dfrac{2}{\pi(1+\omega^2)} & \forall\, \omega \ge 0 \\ 0 & \forall\, \omega < 0 \end{cases}$$

Claim:

X, Y: independent

$Z = X + Y$

$\Rightarrow \quad f_Z = f_X * f_Y$

By induction:

$$\left[Z = \sum_i^n X_i\right] \Rightarrow \left[f_Z = f_{X_1} * \cdots * f_{X_n}\right]$$

Proof:

$$\begin{aligned}
F_Z(z) &= P(X+Y \le z) = P(X \le z - Y) \\
&\overset{TP}{=} \int P(X \le z - Y \mid Y = y) \cdot f_Y(y)\, dy \\
&\overset{S}{=} \int P(X \le z - y \mid Y = y) \cdot f_Y(y)\, dy \\
&\overset{ind}{=} \int P(X \le z - y) \cdot f_Y(y)\, dy
\end{aligned}$$

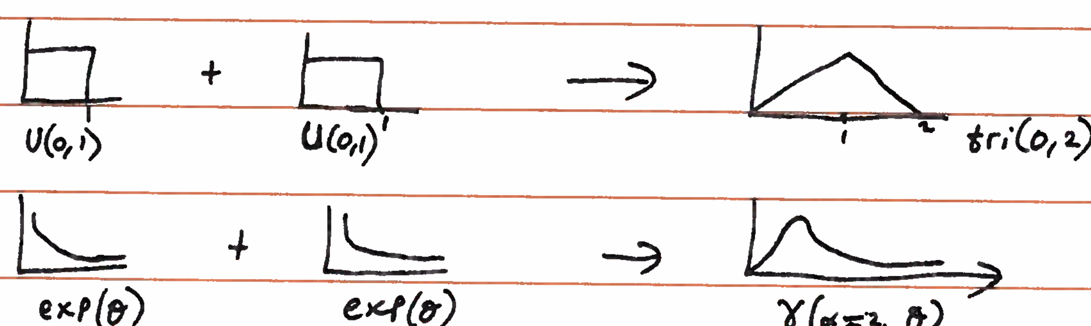
$$F_Z(z) = \int F_X(z-y)\, f_Y(y)\, dy$$

$$\begin{aligned}
\Rightarrow \quad f_Z(z) &= \frac{\partial}{\partial z}\left[\int F_X(z-y) \cdot f_Y(y)\, dy\right] \\
&= \int \frac{\partial F_X(z-y)}{\partial z} \cdot f_Y(y)\, dy \qquad \text{[Leibnitz's Integral Rule]}
\end{aligned}$$

$$\boxed{f_Z(z) = \int f_X(z-y)\, f_Y(y)\, dy}$$

i.e. $\quad f_Z = f_X * f_Y$

e.g.



e.g. $f_{XY}(x,y) = \exp[-(x+y)]$

$Z = \frac{X}{X+Y}$ $\quad [\Rightarrow Z \in (0,1)]$

$f_Z(z) = ?$

Ans: $F_Z(z) = P(Z \le z) = P\left(\frac{X}{X+Y} \le z\right) = P(X \le z(X+Y))$

$= P((1-z)X \le zY)$

$F_Z(z) \overset{T_P}{=} \int P\left(X \le \frac{z}{1-z} Y \,\middle|\, Y=y\right) \cdot f_Y(y)\, dy$

$f_{XY} = f_X \cdot f_Y = \exp(-x) \cdot \exp(-y)$

$\Rightarrow X \perp\!\!\!\perp Y \quad \Rightarrow P\left(X \le \frac{z}{1-z} y \,\middle|\, Y=y\right) = P\left(X \le \frac{zy}{1-z}\right)$

$\Rightarrow F_Z(z) = \int F_X\left(\frac{zy}{1-z}\right) \cdot f_Y(y)\, dy$

$= \int_0^\infty \left[1 - \exp\left(\frac{-zy}{1-z}\right)\right] \cdot \exp(-y)\, dy$

$= \int_0^\infty \exp(-y) - \exp[-y/(1-z)]\, dy$

$= 1 - \left[-(1-z) \cdot \exp\left(\frac{-y}{1-z}\right)\right]\Big|_0^\infty$

$= 1 - (1-z)$

$F_Z(z) = z$

$\therefore \boxed{f_Z(z) = \begin{cases} 1 & z \in [0,1) \\ 0 & z \notin (0,1) \end{cases}}$

$Z \sim U(0,1)$

e.g. a :

$X, Y \sim N(0, \sigma^2)$, independent

$Z = X^2 + Y^2$ [i.e $Z \geq 0$]

$f_Z(z) = ?$

Ans :

$$F_Z(z) = P(Z \leq z)$$
$$= P(X^2 + Y^2 \leq z)$$
$$= \iint_{C_z} f_{XY}(x, y)\, dx\, dy$$
$$C_z = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq z\}$$
$$= \iint_{C_z} \frac{1}{2\pi\sigma^2} \exp\left(-\left(\frac{x^2+y^2}{2\sigma^2}\right)\right) dx\, dy$$

$r^2 = x^2 + y^2$; $dx\, dy = r\, dr\, d\theta$

$C_z = \{(r,\theta) \mid r^2 \leq z\} = \{(r,\theta) \mid r \leq \sqrt{z}\}$

$$\Rightarrow F_Z(z) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\sqrt{z}} \frac{1}{\sigma^2} \exp\left(-r^2/2\sigma^2\right) r\, dr\, d\theta$$
$$F_Z(z) = 2\pi/2\pi \cdot 1/\sigma^2 \int_0^{\sqrt{z}} r \cdot \exp\left(-r^2/2\sigma^2\right) dr$$

$y = r^2/2\sigma^2 \Rightarrow dy = r/\sigma^2\, dr$; $r = \sqrt{z} \Rightarrow y = z/2\sigma^2$

$$\Rightarrow F_Z(z) = \int_0^{z/2\sigma^2} \exp(-y)\, dy$$
$$F_Z(z) = \left[1 - \exp\left(-z/2\sigma^2\right)\right] \cdot u(z)$$

[$\because z \geq 0$]

$$f_Z(z) = dF_Z/dz$$

$$\therefore \boxed{f_Z(z) = 1/2\sigma^2 \cdot \exp\left(-z^2/2\sigma^2\right) \cdot u(z)}$$

i.e $Z \sim \exp(2\sigma^2) = \chi^2(r=2, \theta = 2\sigma^2) = \gamma(1, 2\sigma^2)$

eg 6: $Z = \sqrt{X^2+Y^2}$

$f_Z(z) = ???$

Same argument as in (2a) but $\tilde{C}_z = \{(x,y) \mid x^2+y^2 \le z^2\}$

$\Rightarrow \quad F_Z(z) = \iint_{\tilde{C}_z} f_{XY}(x,y)\, dy\, dx$

in polar coordinates:

$\tilde{C}_z = \{(r,\theta) \mid r \le z\}$

$$\Rightarrow \quad F_Z(z) = 1/2\pi\sigma^2 \cdot \int_0^{2\pi} \int_0^{z} \exp(-r^2/2\sigma^2)\, r\, dr\, d\theta$$
$$= 1/\sigma^2 \cdot \int_0^{z} \exp(-r^2/2\sigma^2)\, r\, dr$$

$[y = r^2/2\sigma^2] \qquad r = z \Rightarrow y = z^2/2\sigma^2$

$$\Rightarrow \quad F_Z(z) = \int_0^{z^2/2\sigma^2} \exp(-y)\, dy$$
$$F_Z(z) = 1 - \exp(-z^2/2\sigma^2)$$
$$f_Z(z) = \partial F_Z/\partial z$$
$$\therefore \quad \boxed{f_Z(z) = z/\sigma^2 \cdot \exp(-z/2\sigma^2) \cdot u(z)}$$

$Z \sim \text{Rayleigh}(\sigma)$

ex: $X, Y \sim N(0,1)$, iid

$Z = X/Y$; $f_Z(z) = ?$

Ans:

$$F_Z(z) = P(X/Y \le z)$$
$$\overset{T_P}{=} \int_{-\infty}^{\infty} P(X/Y \le z \mid Y = y) \cdot f_Y(y)\, dy$$

[need to break integral because $\{X/Y \le z\}$ if different if $Y \gtrless 0$]

$$\overset{S, Ind}{=} \int_{-\infty}^{0} P(X \ge zy)\, f_Y(y)\, dy + \int_{0}^{\infty} P(X \le zy)\, f_Y(y)\, dy$$

$$F_Z(z) = \int_{-\infty}^{0} \big(1 - F_X(zy)\big) f_Y(y)\, dy + \int_{0}^{\infty} F_X(zy)\, f_Y(y)\, dy$$

$$\Rightarrow \quad f_Z(z) = \int_{-\infty}^{0} -y \cdot f_X(zy)\, f_Y(y)\, dy + \int_{0}^{\infty} y\, f_X(zy)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} |y|\, f_X(zy) \cdot f_Y(y)\, dy$$
$$= \int_{-\infty}^{\infty} |y| \cdot \frac{1}{2\pi} \exp\left(-\frac{z^2y^2}{2} - y^2/2\right) dy$$

$$= 1/2\pi \int_{-\infty}^{\infty} |y| \cdot \exp\left(-y^2/2\, (1+z^2)\right) dy$$
$$\overset{\text{even fcn}}{=} 1/\pi \int_{0}^{\infty} y \exp\left(-y^2/2\, (1+z^2)\right) dy$$

$r = y^2/2\, (1+z^2)$ $\quad y \in (0,\infty) \Rightarrow r \in (0,\infty)$

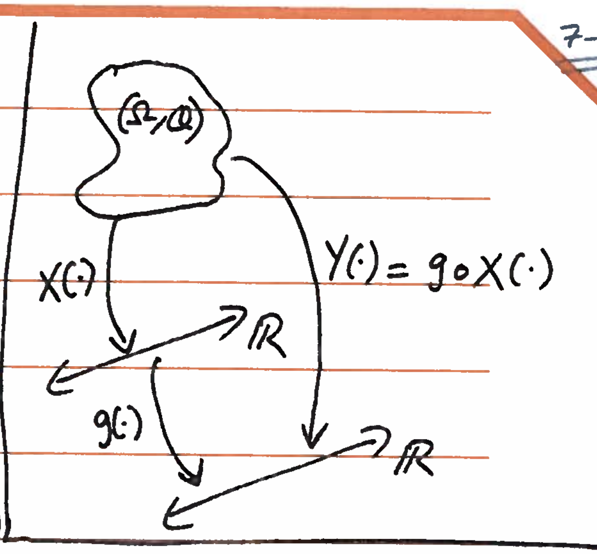
$dr = y(1+z^2) \cdot dy \quad \Rightarrow \quad dy = \dfrac{dr}{y(1+z^2)}$

$$\Rightarrow f_Z(z) = \frac{1}{\pi} \cdot \frac{1}{1+z^2} \cdot \int_{0}^{\infty} \exp(-r)\, dr = \frac{1}{\pi(1+z^2)}$$

i.e $\boxed{Z \sim C(0,1)}$

Tranformations by Formulae:

We can also derive the distribution of $Y = g(X)$ using a formula. The intuition behind the formula is to examine how $g(\cdot)$ "redistributes mass" in terms of $P(A)$



$$\{\omega \in \Omega \mid Y(\omega) \in A_Y\} = \{\omega \in \Omega \mid g \circ X(\omega) \in A_Y\} = \{\omega \in \Omega \mid X(\omega) \in g^{-1}(A_Y)\}$$

$$\Rightarrow P(Y \in A_Y) = P(g \circ X \in A_Y)$$

$$= P(X \in g^{-1}(A_Y))$$

i.e $P_Y(A_Y) = P_X(g^{-1}(A_Y))$

Typically g is not 1-1/onto/simple/linear

So $g^{-1}(A_Y) = \bigcup_i A_{x_i}$ $\quad \{A_{x_i}\}_i$ disjoint

A_{x_i} represent the distinct disjoint solution sets of the equation $y = g(x)$

$$\Rightarrow P_Y(A_Y) = \sum_i P_X(A_{x_i})$$

Recall that Borel sets (like intervals $[y, y+dy]$) completely specify P_Y. So suppose WLOG $A_Y = [y, y+dy]$.

$$\Rightarrow P_Y(A_Y) = F_Y(y+dy) - F_Y(y) = \sum_i [F_X(x_i + dx_i) - F_X(x_i)]$$

$[dy, dx_i \to 0]$ $\approx\!> f_Y(y)|dy| = \sum_i f_X(x_i)|dx_i|$

i.e $P_Y(A_Y) \simeq f_Y(y)|dy| = \sum_i f_X(x_i)|dx_i|$

$$|dy| = |g'(x)|\,|dx|$$

i.e $$\boxed{f_Y(y) = \sum_i f_X(x) \cdot \left|\frac{dx}{\partial y}\right| \Bigg|_{x=x_i}}$$

or $$\boxed{f_Y(y) = \sum_i \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=x_i}}$$

where $\{x_i\}_i$ are the distinct roots of the equation $g(x) = y$. x_i are in terms of y $\forall i$

For multidimensional transformations $\vec{Y} = g(\vec{X})$, we get

$$\boxed{f_{\vec{Y}}(\vec{y}) = \frac{f_{\vec{X}}(\vec{x})}{|\det(\partial\vec{y}/\partial\vec{x})|}\Bigg|_{\vec{x}=\vec{x}_i}}$$

e.g: [$Z \sim C(0,1)$, $W = |Z|$ example again]

$$f_W(\omega) = \sum_i f_Z(z)/g'(z) \Big|_{z=z_i}$$

$W = g(Z) = |Z| \Rightarrow$ roots of $\omega = g(z) = \{-\omega, \omega\}$

$$g'(z) = \begin{cases} 1 & z > 0 \\ -1 & z < 0 \end{cases}$$

$$\Rightarrow f_W(\omega) = f_Z(\omega)\Big/|g'(z)|\Big|_{z=\omega} + f_Z(-\omega)\Big/|g'(z)|\Big|_{z=-\omega}$$

$$= f_Z(\omega) + f_Z(-\omega) = 2 f_Z(\omega)$$

$$f_W(\omega) = \frac{2}{\pi(1+\omega^2)} \quad \forall\ \omega \geq 0$$

eg $Y = g(X) = X^2$; $X \sim N(0, \sigma^2)$

$y = g(x) \Rightarrow x \in \{-\sqrt{y}, \sqrt{y}\}$

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-x^2/2\sigma^2\right)$

$g'(x) = 2x$

$$\Rightarrow f_Y(y) = \left.\frac{f_X(x)}{|2x|}\right|_{x=\sqrt{y}} + \left.\frac{f_X(x)}{|2x|}\right|_{x=-\sqrt{y}}$$

$$= \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(-\sqrt{y})}{2\sqrt{y}}$$

$$\Rightarrow f_Y(y) = 1/2\sqrt{y} \cdot 2 f_X(\sqrt{y})$$

$$f_Y(y) = 1/\sqrt{y} \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-y/2\sigma^2\right) u(y)$$

eg: $Y = g(X) = \exp(X)$; $X \sim N(\mu, \sigma^2)$ $[\Rightarrow Y \geq 0]$

$g'(x) = \exp(x)$; $y = g(x) \Rightarrow x \in \{\ln y\}$

$$f_Y(y) = \left.\frac{f_X(x)}{|\exp(x)|}\right|_{x=\ln y} = \frac{f_X(\ln y)}{|\exp(\ln y)|} = \left.\frac{f_X(\ln y)}{|y|}\right|$$

$$f_Y(y) = \frac{1}{y\sigma \cdot \sqrt{2\pi}} \cdot \exp\left(\frac{-1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2\right) u(y)$$

i.e $\boxed{Y \sim \text{lognormal}(\mu, \sigma^2)}$